

The Simple Economics of Optimal Persuasion*

Piotr Dworczak

Northwestern University

Giorgio Martini

Stanford University

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Abstract

We propose a price-theoretic approach to Bayesian persuasion by establishing an analogy between the Sender's problem and finding Walrasian equilibria of a Persuasion Economy. The Sender, who acts as a consumer, purchases posterior beliefs at their prices using the prior distribution as her endowment. A single firm has the technology to garble the state. Welfare theorems provide a verification tool for optimality of a persuasion scheme and characterize the structure of prices that support the optimal solution. This approach yields a tractable solution method for persuasion problems in which the Sender's utility depends only on the expected state.

Keywords: Bayesian persuasion, information design, mean-preserving spreads, monotone partitions, prices, Walrasian equilibrium.

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1 Introduction

Bayesian persuasion has become a canonical model of communication with commitment power following [Kamenica and Gentzkow \(2011\)](#).¹ However, the standard approach to optimal persuasion based on concavification of the value function has limited power when the state space is large. The concavification method alone is typically not sufficient to characterize the optimal signal, and fails to provide intuition for the structure of the underlying persuasion scheme. To overcome this difficulty, we develop a price-theoretic approach to Bayesian persuasion under the assumption that the Sender’s preferences only depend on the mean of posterior beliefs.

We show that the Sender acts as a consumer who purchases posterior means at their prices, subject to a budget constraint, using the prior distribution as her endowment. Prices are determined in a Walrasian equilibrium of a “Persuasion Economy” which features posterior means as “goods”, the Sender as the only consumer, and a single firm. The firm has the technology to garble the state, or “merge the goods”, and maximizes profits.

We prove analogues of the two welfare theorems for this economy. In [Theorem 1](#), we show that competitive equilibria are efficient (hence, since there is only one agent, optimal). This provides a verification tool for optimality of a candidate solution to the Sender’s problem. In [Theorem 2](#), under mild regularity assumptions we show that the Walrasian approach is also necessary: given any optimal persuasion scheme, it is always possible to find a price schedule that supports it as an equilibrium allocation.

Using the analogy to a Walrasian economy, we derive joint restrictions on equilibrium allocations and prices which provide insights about the structure of the optimal persuasion scheme. We show that the technological constraints of the persuasion economy imply that the price schedule must be convex. Moreover, in regions where the price function is strictly convex, the state must be fully revealed. Pooling only happens in regions where prices are linear. These restrictions narrow down the set of candidate solutions to a relatively low-dimensional class in which the optimal solution can often be found using a simple graphical method.

Persuasion mechanisms used in practice often have a simple structure: information is either fully revealed or adjacent types are pooled together (e.g. coarse ratings used by bond rating agencies). In [Theorem 3](#), we use our approach to derive a necessary and sufficient condition on the Sender’s utility function under which a monotone partitional signal is optimal for any prior distribution of the state.

We use the price-theoretic approach to solve two examples. In the first application, an agent must be persuaded to exert effort on a project. The agent is rewarded with a fraction of the value of the project but only the principal knows how much the project is worth if successful. We prove that the principal should disclose the project’s value when it is low, and pool high realizations into the lowest signal that induces maximal effort. In the second application, a financial analyst who possesses private information on the profitability of a risky asset wants to persuade

¹Prior works containing important contributions to the literature on communication under commitment include [Aumann and Maschler \(1995\)](#), [Calzolari and Pavan \(2006\)](#), [Ostrovsky and Schwarz \(2010\)](#) and [Rayo and Segal \(2010\)](#).

an agent to invest in it; the optimal persuasion mechanism has a tractable structure in which the informativeness of the recommendation depends on the agent’s degree of risk aversion. In both applications, a simple graphical analysis combined with our verification result is enough to characterize the optimal persuasion mechanism.

We show that our methods extend beyond the simple case in which a single Sender’s utility depends solely on the one-dimensional posterior mean. In Section 7.1, we consider a model of competition in persuasion. There are multiple Senders with access to the same information, and they disclose signals simultaneously. Using our method, we characterize the set of equilibrium distributions of posterior means. In Section 7.2, we show that our verification result extends immediately to the multidimensional setting, where a (single) Sender’s preferences depend only on the vector of posterior means.

In Section 8, we demonstrate that the analogy to Walrasian equilibrium holds for a general Bayesian persuasion problem, with no restrictions on the objective function of the Sender. In this case, prices are defined on the space of posterior beliefs rather than posterior means. We show that supporting prices provide a dual characterization of the Sender’s value function, complementary to concavification, and can be seen as a “convex closure” of the utility function. We apply this approach to solve an example in which the Sender’s utility depends on a continuous state through two moments of the posterior distribution.

We are not the first to study Bayesian persuasion in the case where payoffs only depend on the posterior mean. Along with their analysis of the general model, [Kamenica and Gentzkow \(2011\)](#) apply the concavification approach to the same setting as ours. The concavification of the value function over posterior means reveals whether the Sender benefits from persuasion, but does not explicitly characterize the value of the problem or the optimal signal. In contrast, our approach directly establishes the structure of the optimal signal. [Kolotilin \(2018\)](#) uses duality theory to characterize the optimal persuasion scheme in a related model, and introduces the idea of prices for messages in the context of an example. The graphical approach introduced in [Gentzkow and Kamenica \(2015\)](#) gives further insights about the structure of the problem by relating it to the choice of a convex function lying between those generated by no disclosure and full disclosure.

The rest of the paper is organized as follows. The next section sets up the model. In Section 3, we prove our main results, and describe the connection to Walrasian equilibria. In Section 4, we analyze the structure of the optimal solution and equilibrium prices. Section 5 provides a necessary and sufficient condition for optimality of monotone partitional signals. In Section 6, we work through two applications of our methods. In Section 7 we discuss extensions to competition in persuasion and to multidimensional persuasion. In Section 8 we discuss the general persuasion problem and the connection of our approach to concavification. Proofs are collected in the Appendix.

2 Model

The state of nature is the realization of a real-valued random variable X with a cumulative distribution function F .² We assume that X has realizations in some non-degenerate bounded interval $[\underline{x}, \bar{x}]$, and its support includes the endpoints $\underline{x}$ and $\bar{x}$. Without loss of generality, we normalize $\underline{x} = 0$ and $\bar{x} = 1$.

A Sender chooses an information structure, that is, a measurable mapping $\pi : [0, 1] \rightarrow \Delta(S)$, for some signal space S .³ Given an information structure π , and the prior belief F , every signal realization $s \in S$ induces a posterior belief over the distribution of X . We assume that the Sender's utility depends on posterior beliefs only through the posterior mean. Formally, there exists a measurable function $u : [0, 1] \rightarrow \mathbb{R}$ such that $u(x)$ is the ex-post utility of the Sender when the induced posterior mean is x .

Under this assumption, the expected value of an information structure π depends only on the distribution of posterior means that it induces. It is thus natural to optimize over distributions of posterior means directly. Given the prior F , a distribution of posterior means G is induced by some information structure if and only if F is a mean-preserving spread of G (Blackwell, 1953; Kolotilin, 2018; Gentzkow and Kamenica, 2015).⁴ The Sender's problem is thus

$$\max_G \int_0^1 u(x) dG(x) \tag{2.1}$$

subject to the constraint that F is a mean-preserving spread of G .

The above problem for the Sender arises in the following Bayesian persuasion model: There is a Receiver who holds a prior F , observes signal s , and takes an action. The optimal action only depends on the expected state. The Sender has commitment power, and her preferences over Receiver's actions and states are affine in the state (for example, are state-independent). The Receiver is not part of the formal model but her decision problem influences the shape of the Sender's objective function u . Whenever we refer to the Receiver, we mean that u is derived from the above persuasion model.

While we focus on persuasion as the main application of our methods, we note that the problem (2.1) can also arise in various other contexts, for example, in certain models of costly information acquisition.

²We assume throughout that there exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which X is defined. The explicit probability space plays no further role in the analysis.

³The signal space S is endowed with a relevant σ -field, and π is measurable with respect to the σ -field on $[0, 1]$ with respect to which X is a random variable (which can be taken to be, for example, the Borel σ -field).

⁴There are many equivalent definitions of mean-preserving spreads (see e.g. Müller and Stoyan, 2002). We will use two: F is a mean-preserving spread of G if $\int_0^1 v(x) dF(x) \geq \int_0^1 v(x) dG(x)$ for all convex v ; equivalently, if $\int_0^t F(x) dx \geq \int_0^t G(x) dx$ for all $t \in [0, 1]$ and $\int_0^1 F(x) dx = \int_0^1 G(x) dx$.

3 A price-theoretic approach to persuasion

Our first result provides a way to verify optimality of a candidate solution G by means of an auxiliary function p .

Theorem 1. *If there exist a cumulative distribution function G and a convex function $p : [0, 1] \rightarrow \mathbb{R}$, with $p(x) \geq u(x)$ for all $x \in [0, 1]$, that satisfy*

$$\text{supp}(G) \subseteq \{x \in [0, 1] : u(x) = p(x)\}, \quad (3.1)$$

$$\int_0^1 p(x) dG(x) = \int_0^1 p(x) dF(x), \text{ and} \quad (3.2)$$

$$F \text{ is a mean-preserving spread of } G, \quad (3.3)$$

then G is a solution to problem (2.1).

Proof. See Appendix A.1.

We will be using Theorem 1 primarily as a tool to solve problem (2.1). However, the result also provides a useful interpretation of the optimal persuasion scheme as a Walrasian equilibrium allocation. The auxiliary function p describes equilibrium prices for posterior means. The Sender is a consumer endowed with F who selects a bundle G to maximize utility subject to a budget constraint, similar to conditions (3.1) – (3.2). A single firm transforms states into posterior means using a technology that satisfies the mean-preserving-spread condition (3.3). We formalize the analogy next.

3.1 The “Persuasion Economy”

We begin by describing the “Persuasion Economy”. Let $[0, 1]$, the set of possible posterior means, be the space of goods. A bundle of goods, possibly including negative quantities, is denoted by a right-continuous function $G : [0, 1] \rightarrow \mathbb{R}$, where $G(x)$ is the total mass of goods in $[0, x]$. When the bundle includes non-negative amounts of each good and the total mass of goods is 1, G is a cumulative distribution function. Let M denote the set of possible bundles.⁵

There is a single consumer with utility $u(x)$ for one unit of good x . Utility is linear in quantity and additively separable across goods: utility of bundle G is $\int_0^1 u(x) dG(x)$. The consumer can only consume non-negative amounts of each good, so the consumption set $M^+ \subset M$ consists of non-negative bundles: G must be non-decreasing. The consumer is initially endowed with a bundle F of goods.

⁵Formally, M is the set of signed, countably additive, regular, finite-variation measures on $[0, 1]$ with the Borel σ -algebra. With slight abuse of notation, we identify a measure $\mu \in M$ with its cumulative distribution $G(x) = \mu([0, x]) = \int_0^x d\mu(z)$.

There is a single competitive firm that chooses a production plan Z from the following production possibility set:

$$\mathcal{Z} = \left\{ Z \in M : \int_0^x Z(t)dt \leq 0, \forall x \in [0, 1], \int_0^1 Z(t)dt = 0, Z(1) = 0 \right\}.$$

$\mathcal{Z}$ describes the technology of taking mean-preserving contractions. That is, the firm takes “states” as inputs, and produces posterior means as outputs. For example, a unit of good x_1 and a unit of good x_2 can be used to produce two units of good $(x_1 + x_2)/2$. The additional condition $Z(1) = 0$ says that the total mass of inputs is equal to the total mass of outputs, and can be interpreted as no free disposal (this rules out uninteresting equilibria). The consumer is the sole shareholder of the firm.

Definition. (G, Z, p) is a *Walrasian equilibrium* if

1. The consumer maximizes utility given the budget constraint:

$$G \in \operatorname{argmax}_{\tilde{G} \in M^+} \int_0^1 u(x)d\tilde{G}(x) \text{ subject to } \int_0^1 p(x)d\tilde{G}(x) \leq \int_0^1 p(x)dF(x) + \pi, \quad (3.4)$$

where π is the dividend from the firm.

2. The firm maximizes profits $\pi = \int_0^1 p(x)dZ(x)$:

$$Z \in \operatorname{argmax}_{\tilde{Z} \in \mathcal{Z}} \int_0^1 p(x)d\tilde{Z}(x). \quad (3.5)$$

3. Markets clear:

$$G = F + Z. \quad (3.6)$$

We argue that (G, Z, p) is a Walrasian equilibrium if and only if (G, p') satisfies conditions (3.1)–(3.3) of Theorem 1, where p' is an affine transformation of p (prices can be rescaled without affecting equilibrium properties).

1. In order for problem (3.5) to admit a solution, prices must be convex (otherwise, the firm can scale up production and achieve infinite profits). Furthermore, when prices are convex, the firm cannot make strictly positive profits (and can always achieve zero profits by not producing anything). Thus, the firm breaks even in equilibrium. By market clearing, the firm must produce $Z = G - F$. The zero-profit condition corresponds to (3.2).
2. In the consumer’s problem in the economy, let λ be the Lagrange multiplier on the consumer’s budget constraint. Due to linearity, maximizing the Lagrangian implies that $\operatorname{supp}(G) \subseteq \operatorname{argmax}_x \{u(x) - \lambda p(x)\}$. Moreover, the consumer chooses a bundle corresponding to a cdf in equilibrium: Market clearing and the no-free-disposal condition for the firm imply $G(1) = 1$. By letting $\beta \equiv \max_x \{u(x) - \lambda p(x)\}$, we can rescale prices to $p' = \lambda p + \beta$, and conclude that

(3.1) and $p' \geq u$ both hold. Conversely, if we start with (G, p) satisfying (3.1)–(3.3), then G is a solution to the consumer’s problem because the Lagrangian is maximized (with $\lambda = 1$), and the consumer’s budget constraint is satisfied by (3.2).

3. Market clearing is implicit in equations (3.1)–(3.3). Condition (3.3) corresponds to the technological constraint of the firm.

3.2 Welfare Theorems for the Persuasion Economy

We can now interpret Theorem 1 as the First Welfare Theorem for the Persuasion Economy. If (G, Z, p) is an equilibrium, then G is Pareto efficient.⁶ Efficiency in a one-consumer economy means that the allocation maximizes the consumer’s welfare. Hence, G is optimal for the Sender.

Conversely, we might expect the Second Welfare Theorem to also hold for the Persuasion Economy: Do there always exist prices that support an optimal solution as an equilibrium allocation? We give a positive answer under mild regularity conditions on u .

Definition 1. Function u is *regular* if:

- (i) u is upper semi-continuous with at most finitely many one-sided jump discontinuities at interior points $y_1, \dots, y_k \in (0, 1)$, and has bounded slope (is Lipschitz continuous) in each (y_i, y_{i+1}) , with $y_0 = 0$ and $y_{k+1} = 1$;⁷
- (ii) There exists a finite partition of $[0, 1]$ into intervals such that u is either strictly convex, strictly concave, or affine on each interval in that partition.⁸

Theorem 2. Suppose that u satisfies condition (i) of regularity. Then there exists an optimal solution G , and for every optimal solution G , there exists a convex and continuous $p : [0, 1] \rightarrow \mathbb{R}$ such that the pair (G, p) satisfies conditions (3.1)–(3.3).

Proof. See Appendix A.2.

Because the Persuasion Economy has infinitely many goods, standard proofs of the Second Welfare Theorem cannot be adopted easily. Instead, to prove Theorem 2, we use duality techniques from the literature on optimization with stochastic dominance constraints (Dentcheva and Ruszczyński, 2003). Because problem (2.1) does not satisfy the qualification constraint imposed by Dentcheva and Ruszczyński, we construct a sequence of problems approximating (2.1) in which the qualification constraint holds, and show the appropriate convergence properties. This, however,

⁶Even when u is non-positive (so that consumer’s preferences are not necessarily locally insatiable) the First Welfare Theorem holds because the consumer always exhausts her budget in equilibrium. If the budget constraint were slack, then the firm would be making negative profits, which cannot happen in equilibrium.

⁷In the Bayesian persuasion model, upper semi-continuity could be derived as a consequence of the primitives under the additional assumption that the Receiver chooses the Sender-preferred action when indifferent.

⁸For example, $f(x) = \sin(\frac{1}{1-x})(1-x)^2$ for $x < 1$ and $f(1) = 0$ is continuous and has a bounded slope but is not regular because it switches from concavity to convexity infinitely many times. This regularity condition is not needed for most results but simplifies the analysis of supporting prices. We would like to thank an anonymous referee for alerting us to a problem with the definition of regularity used in a previous version of this paper.

only gives us existence of one solution G supported by a price function p (the limit of the sequence). To prove existence of supporting prices for *any* solution G , we use the following observation (proven in Appendix A.2).

Fact 1. *If G and G' are both solutions to problem (2.1), and (G, p) satisfy conditions (3.1)–(3.3), then (G', p) also satisfy (3.1)–(3.3).*

Fact 1 should not come as a surprise because the Persuasion Economy is quasi-linear: The consumer's utility is linear in quantities, so there exists a numeraire good that can be treated as money. Gul and Stacchetti (1999) show that *any* equilibrium allocation in a quasi-linear economy is supported by *any* equilibrium price function, a direct analog of Fact 1. Due to the product structure of allocation and prices, we can strengthen Theorem 2 in the following way.

Corollary 1. *If u satisfies condition (i) of regularity, then there exists a convex and continuous $p : [0, 1] \rightarrow \mathbb{R}$ such that G is optimal if and only if it satisfies (3.1)–(3.3).*

In light of Corollary 1, one could hope that the supporting price function can be characterized independently of the optimal solution G . We show next that this is indeed the case.

Proposition 1. *Suppose that u satisfies condition (i) of regularity. If a price function p^* solves*

$$\min_p \int_0^1 p(x) dF(x) \text{ subject to } p \text{ being convex and } p \geq u, \quad (3.7)$$

then p^ is an equilibrium price function, and in particular satisfies (3.1)–(3.3) with any optimal solution G . Conversely, any price function satisfying (3.1)–(3.3) with some G is a solution to (3.7).*

Proof. See Appendix A.3. □

By Proposition 1, the equilibrium price function minimizes the value of the endowment among all feasible price functions (the constraint $p \geq u$ is needed to obtain the particular normalization of the price function used in Theorem 1). The intuition follows again from looking at the Persuasion Economy. As one of the proofs of existence of Walrasian equilibria reveals, we can think of prices as set by a mythical player, the Walrasian auctioneer, who aims to maximize the value of the net trade. In the Persuasion Economy, the net trade is $G - F - Z$. Because $\int_0^1 p(x) dZ(x) = 0$ and $\int_0^1 p(x) dG(x)$ is proportional to $\int_0^1 u(x) dG(x)$ when the firm and the consumer are maximizing, in our economy the auctioneer focuses on minimizing the value of the endowment F .

Proposition 1 and Corollary 1 provide a solution method for the Sender's problem: First solve (3.7) to find p , then find G that satisfies conditions (3.1)–(3.3). While this is always a valid approach under the regularity assumptions on u , solving (3.7) is non-trivial. We derive an alternative approach based on joint restrictions on (G, p) in Section 4.

3.3 Relationship to other duality methods

Kolotilin (2018) is the first to introduce ideas from linear programming to persuasion problems. He provides a strong duality theorem for a model of persuasion in which the Receiver is privately informed and chooses between two actions.⁹ He also studies a case of his model that is mathematically equivalent to our setting. Our Theorems 1 and 2 bear similarity to the strong duality result of Kolotilin but are different in the following way. Kolotilin uses an alternative representation of the persuasion problem, and hence his Lagrange multiplier is not equivalent to our price function. Our formulation in terms of the price function permits the interpretation of the persuasion problem as a Walrasian equilibrium.

Daskalakis, Deckelbaum, and Tzamos (2017) study the problem of finding a profit-maximizing mechanism for a multi-product monopolist. Their primal problem involves optimizing over convex functions, and they show that a suitable dual formulation obtains using mean-preserving spreads; our setting is mirrored, with mean-preserving spreads in the primal and convex functions in the dual. The approach we use to prove strong duality (Theorem 2) is however very different: Daskalakis, Deckelbaum, and Tzamos use methods from optimal transport, whereas we use results from infinite-dimensional linear programming. As Daskalakis, Deckelbaum, and Tzamos note, it is often difficult to establish existence of interior points in the feasible set, which is necessary for strong duality; we bypass this difficulty by studying the limit of a sequence of perturbed problems where the interior is non-empty. Whether optimal transport methods can be applied to persuasion problems is an interesting open question.

4 Structure of prices

We first show how to construct supporting prices for the cases in which the solution to the persuasion problem is known and takes a simple form. We then characterize the general structure of supporting prices, which allows us to use Theorem 1 to solve non-trivial persuasion problems.

If u is concave — and more generally if there exists an affine function q such that $q(x) \geq u(x)$ for all x and $q(\mathbb{E}X) = u(\mathbb{E}X)$ — then pooling (revealing nothing about X) is a solution to the Sender’s problem. To see this using our results, it is enough to take $p \equiv q$. Since p is affine, condition (3.2) is equivalent to equality of unconditional means of F and G . The distribution G that puts all mass on $\mathbb{E}X$ also satisfies conditions (3.1) and (3.3), and hence is optimal by Theorem 1.

If u is convex, then full revelation (always revealing X) is a solution to the Sender’s problem. In this case we can take $p \equiv u$ and $G \equiv F$ for which conditions (3.1)–(3.3) hold trivially.

Another simple example where our methods can be applied is when the Receiver chooses one of two actions. Suppose that the Receiver takes the Sender-preferred action if and only if her posterior mean is greater than or equal to x_0 (so that indifferences are broken in Sender’s favor). We normalize Sender’s utility to 0 and 1 for the two actions. This example can be solved by other

⁹ Related models of persuasion with a privately informed receiver (but not using duality techniques) are considered by Kolotilin, Mylovanov, Zapechelnyuk, and Li (2017) and Guo and Shmaya (2017).

methods (see both [Gentzkow and Kamenica, 2015](#) and [Ivanov, 2015](#)).

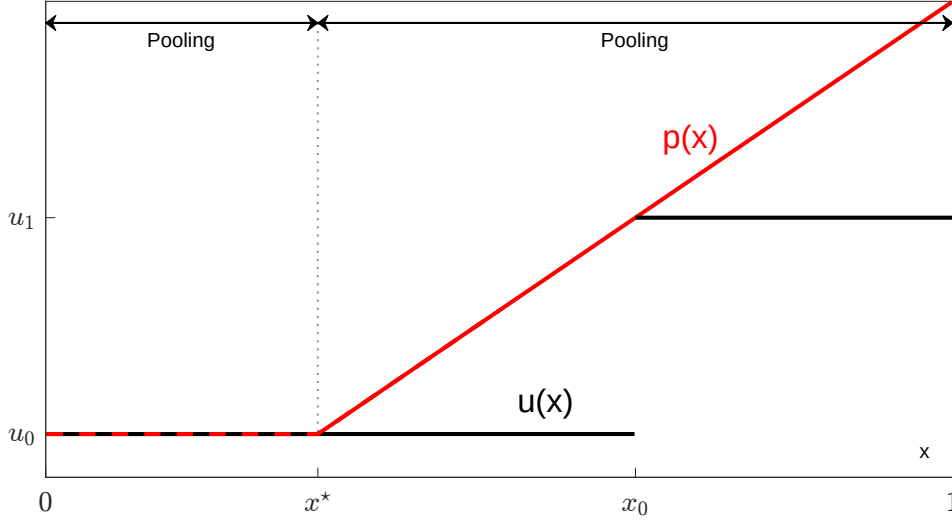


Figure 1: Two-action case

Corollary 2. *Assume that F has no atoms. Let u be the non-decreasing step function $u(x) = \mathbb{1}(x \geq x_0)$. If $\mathbb{E}X \geq x_0$, then it is optimal to reveal nothing. If $\mathbb{E}X < x_0$, then it is optimal to reveal whether x is below or above x^* , where x^* satisfies $\mathbb{E}[X|X \geq x^*] = x_0$.*

Proof. If $\mathbb{E}X \geq x_0$, then $p \equiv 1$ and $G(x) = \mathbb{1}(x \geq \mathbb{E}X)$ trivially satisfy conditions (3.1)–(3.3), hence no revelation is optimal. Now assume $\mathbb{E}X < x_0$. Let x^* be defined by $\mathbb{E}[X|X \geq x^*] = x_0$. Consider the piece-wise affine p and cdf G given respectively by

$$p(x) = \begin{cases} 0 & 0 \leq x < x^* \\ \frac{x - x^*}{x_0 - x^*} & x^* \leq x \leq 1 \end{cases} \quad \text{and} \quad G(x) = \begin{cases} 0 & 0 \leq x < \mathbb{E}[X|X < x^*] \\ F(x^*) & \mathbb{E}[X|X < x^*] \leq x < x_0 \\ 1 & x_0 \leq x \leq 1 \end{cases}.$$

The function p is convex by construction. Condition (3.1) holds because u and p coincide for $x \leq x^*$ and $x = x_0$. Condition (3.2) is satisfied because p is piece-wise affine, and the conditional means of F and G are equal in both regions, by construction. By the way we defined G , F is a mean-preserving spread of G , which verifies condition (3.3). Thus, by Theorem 1, G is optimal. $\square$

In all previous examples the optimal G takes a simple form, and it is straightforward to use Theorem 1 to verify optimality. We now prove that conditions (3.1)–(3.3) impose tight joint restrictions on (G, p) which greatly reduce the set of candidate solutions and supporting prices.

Proposition 2. *Suppose that F has full support and no mass points. Let p be a convex and continuous function that satisfies conditions (3.1)–(3.3) for some distribution G . Then, for any interval $[a, b] \subseteq [0, 1]$:*

- (i) if p is strictly convex on $[a, b]$, then $p(x) = u(x)$ and $G(x) = F(x)$, for all $x \in [a, b]$; or,
- (ii) if p is affine on $[a, b]$, and $[a, b]$ is a maximal interval on which p is affine, then $G(a) = F(a)$, $G(b) = F(b)$, $\int_a^b t dG(t) = \int_a^b t dF(t)$, and $p(c) = u(c)$ for at least one $c \in [a, b]$.

Furthermore, if u is regular, then there exists a coarsest partition $0 = x_0 < x_1 < \dots < x_{n+1} = 1$ of $[0, 1]$ such that on each $[x_i, x_{i+1}]$ the conclusion of either (i) or (ii) holds.

Proof. See Appendix A.4.

The key property of the supporting price function can be understood in the context of the Persuasion Economy. In regions where the price function is strictly convex, the firm cannot engage in any non-trivial production as this would yield losses (given convexity of prices and the technological constraints). Thus $dF = dG$ (so that $dZ = 0$) in these regions which corresponds to full disclosure of the state. Non-trivial production can only take place in intervals where the price function is affine. On such intervals, the production of the firm is only constrained by the mean-preserving spread and no-free-disposal conditions.

Proposition 2 implies that it is enough to verify the mean-spread condition (3.3) separately for every maximal interval in which p is affine (the condition holds automatically conditional on realizations in intervals where p is strictly convex). Moreover, whenever $p(x) > u(x)$ for all $x \in [a, b]$, then p is piece-wise affine with at most one kink in $[a, b]$. Indeed, such $[a, b]$ can intersect at most two consecutive intervals $[x_i, x_{i+1}], [x_{i+1}, x_{i+2}]$ because in every interval of the partition, p and u coincide at at least one point.

The conditions listed above often restrict the set of possible prices to a small class. Because the convex price function p can only coincide with u when u is convex, and p is piece-wise affine with at most one kink between regions where $u \equiv p$, for relatively simple u , the set of potential price functions can be parameterized by a low-dimensional parameter. For any p in that set, points where $u \equiv p$ determine the support of G , and the supporting price function can be found by solving for the parameter which yields condition (3.3). We further aid the search for the solution in Section 5 where we provide sufficient conditions for when a monotone partitional signal is optimal. In Section 6, we illustrate the method by applying it to two persuasion problems.

5 Monotone partitional signals

Persuasion mechanisms that are seen in practice often only use *monotone partitional signals*: pooling, if present, is only between adjacent types. For example, many schools only release coarse information on student performance (Ostrovsky and Schwarz, 2010). Bond credit ratings also have a coarse structure (where very fine categories can be interpreted as full disclosure). These signal structures also appear in other models of communication: Crawford and Sobel (1982) show that all equilibria in their model feature monotone partitional signals.

Definition. A distribution of posterior means G is induced by a *monotone partitional signal* if there exists a finite partition of $[0, 1]$ into intervals $\{[x_i, x_{i+1}]\}_{i=1}^k$ such that for each i , either (i) $G \equiv F$ in $[x_i, x_{i+1}]$ (full revelation), or (ii) G puts all mass in $[x_i, x_{i+1}]$ on $\mathbb{E}[X | X \in [x_i, x_{i+1}]]$ (pooling).

We prove that the following definition gives a sufficient and *necessary* condition on the objective function under which a monotone partitional signal is optimal regardless of the underlying prior distribution over states.

Definition 2. A function u is *affine-closed* if there do not exist $0 < x < y < 1$ and an affine function q such that:

1. $u(x) = q(x)$ and $u(y) = q(y)$;
2. $q(z) \geq u(z)$ for all $z \in [x, y]$;
3. $q(z) > u(z)$ for all $z \in \{w\} \cup (x - \epsilon, x) \cup (y, y + \epsilon)$ for some $\epsilon > 0$ and some $w \in (x, y)$.

To provide some intuition for the above definition, note first that adding an affine function to u does not change the optimal solution to our problem. Roughly speaking, an affine-closed function is defined by the property that $u + q$ has at most one local interior maximum for any affine function q . When this is not the case, that is, $u + q$ has two (or more) interior “peaks”, we can find an affine function q' tangent to these peaks that would satisfy properties 1-3. Importantly, an affine-closed function can have local (one-sided) maxima at the endpoints. This intuition is precise when u itself is not affine in any interval. The definition is, however, more permissive when u is locally affine and can therefore have an interval of local maxima.

Convex and concave functions are always affine-closed. More complex examples of affine-closed functions are shown in Figures 2 and 3. In both cases there does not exist an affine function q with properties 1-3 listed in Definition 2. The affine function q_1 in Figure 2 does not satisfy property 1 because points of support must be interior (not 0 or 1). The affine function q_2 that supports u at x and y does not satisfy property 3, because there does not exist $w \in (x, y)$ such that $q_2(w) > u(w)$. Finally, the function in Figure 3 is affine-closed because the affine function q_3 cannot simultaneously satisfy properties 2 and 3, regardless of how we choose the support points x and y .

Theorem 3. *Let u be regular. If u is affine-closed, then for any continuous and full-support prior F there exists an optimal solution G for problem (2.1) which is induced by a monotone partitional signal. Conversely, if u is not affine-closed, then there exists a (continuous) prior F such that no optimal G can be induced by a monotone partitional signal.*

Proof. See Appendix A.5.

In the proof, we use Theorem 2 to generate a solution G and a corresponding multiplier p . Starting from G , we construct a modified distribution of posteriors which is induced by a monotone partitional signal. Using Proposition 2 and the affine-closure property, we show that the multiplier

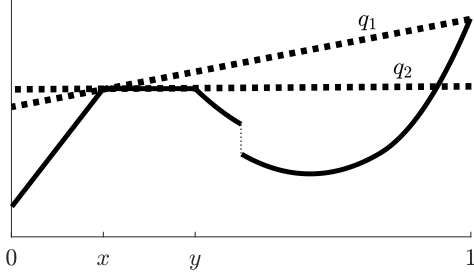


Figure 2: An affine-closed function (solid line) and affine functions q_1 and q_2 (dotted lines)

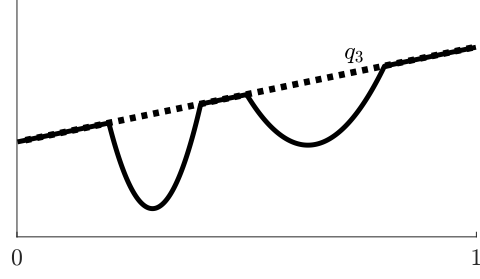


Figure 3: An affine-closed function (solid line) and an affine function q_3 (dotted line)

p still supports the modified distribution, thus proving its optimality by Theorem 1. To prove the converse, we use the violation of affine-closure to construct a distribution F such that the optimal signal cannot have a monotone partitional structure.

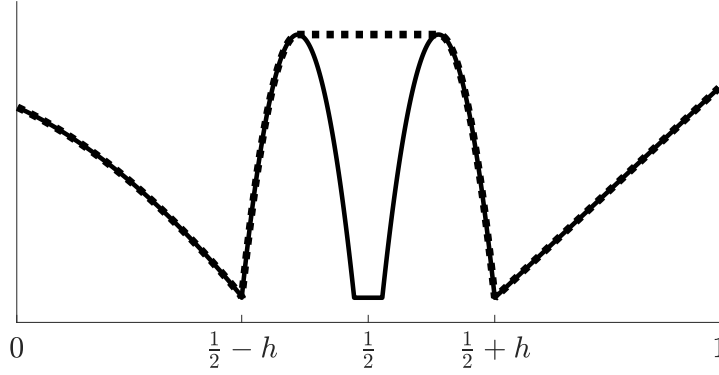


Figure 4: A non-affine-closed function (solid line) and an affine-closed function (dotted line).

The importance of the affine-closure assumption can be understood using Figure 4. Consider the non-affine-closed function (solid line). Suppose that F is the uniform distribution on $[0, 1]$. If $h \leq \frac{1}{4}$ in Figure 4, then by Theorem 1, the optimal posterior distribution of means has two atoms at the two peaks $\frac{1}{2} \pm h$ of the objective function (the multiplier p is a horizontal line tangent at the two peaks). Except for the non-generic case $h = \frac{1}{4}$, this distribution of posterior means cannot be induced by a monotone partitional signal. Consider instead the affine-closed function in Figure 4 (dotted line) as the objective. Although the same signal remains optimal, we can now modify it to obtain a monotone partition: We construct a signal which pools all realizations into one posterior mean $\frac{1}{2}$, and achieves the same payoff as the mixture between points $\frac{1}{2} - h$ and $\frac{1}{2} + h$. The affine closure property implies that the objective function coincides with the (locally affine) price function at the pooled mean $\frac{1}{2}$, allowing an optimal G to put an atom at $\frac{1}{2}$, in line with condition (3.1).

Continuity of F is needed for the first half of the theorem. Consider the extreme case where F puts all mass on two points. Then, there are only two distinct distributions G that are induced by monotone partitions: $G = F$ (full revelation) and $G = \delta_{\mathbb{E}[X]}$ (pooling at the prior mean). It is easy

to construct (affine-closed) u for which neither of these is optimal.¹⁰

The affine-closure property is distinct from the notion of a concave closure (as Figures 2 and 3 illustrate). Nevertheless, one can ask if there exists a well-defined notion of an affine closure $\hat{u}$ of the function u with the property that it produces the same value of the persuasion problem as u but the optimal signal is monotone partitional. The price function p from Theorem 1 satisfies all properties required of $\hat{u}$: it is affine-closed (because it is convex), and it yields the same value of the optimal persuasion problem as u (this follows from convexity of p and conditions (3.1) - (3.2)). The caveat is that p depends on the prior F . It is not possible to define affine closure $\hat{u}$ so that $\hat{u}$ is affine-closed and yields the same value of the persuasion problem as u for *every* F .

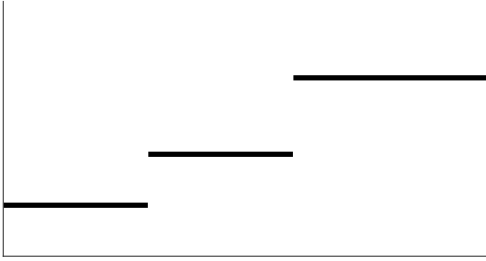


Figure 5: A non-affine-closed function

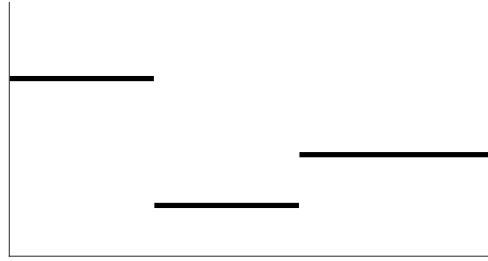


Figure 6: An affine-closed function

One natural example where the affine-closure assumption may not hold is when the Sender has state-independent preferences, and the Receiver chooses one of three actions. As noted in [Gentzkow and Kamenica \(2015\)](#), the optimal signal structure may fail to be monotone partitional in this case. Our method yields a complete characterization of cases when the existence of a monotone partitional signal is guaranteed. When the three actions are ordered, as in Figure 5, u is not affine-closed. By Theorem 3, there exists a prior distribution for which no monotone partitional signal is optimal. On the other hand, when the actions are not ordered (both when the central action is the worst, as in Figure 6, and when it is the best), u is affine-closed and hence there always exists an optimal monotone partitional signal.

Relationship to the literature Several other papers have found sufficient conditions for monotone partitional signals in persuasion models. In the general setting of [Kamenica and Gentzkow \(2011\)](#), [Mensch \(2018\)](#) gives a sufficient condition based on supermodularity of the Sender’s and Receiver’s preferences over action and state. [Ivanov \(2015\)](#) studies a setting in which the preferences of the Sender depend not only on the posterior mean but also on the order of posterior means induced by a signal. Such a setting captures communication problems more general than Bayesian persuasion. The sufficient conditions of both [Mensch](#) and [Ivanov](#) become trivial when projected to our setting – they say that the objective function is convex. Finally, in a setting analogous to ours, [Kolotilin \(2018\)](#) provides a characterization of *interval revelation schemes*. Because interval

¹⁰However, for the case of discrete distributions, one can derive a version of Theorem 3 under a more permissive definition of a monotone partition that permits “splitting” of an atom into two adjacent signals.

revelation schemes, as defined by [Kolotilin](#), belong to the class of monotone partitional signals, his condition, which depends on the prior distribution, is sufficient. We give a prior-independent condition that is both necessary and sufficient.

6 Applications

In this section, we show that the methods developed in the paper can be successfully applied to solve persuasion problems with large action and state spaces. Concurrent work by the first author ([Duffie, Dworczak, and Zhu, 2017](#)) uses our results in a model studying optimal pre-trade transparency of financial over-the-counter markets. Dealers trade with customers who are uninformed about the common cost of providing an asset, and hence face uncertainty over the prices quoted by dealers. Because of costly search for the best price, entry by customers is limited. A social planner decides how much information about the common cost to reveal prior to trading. The distribution of the cost is continuous, resulting in a continuum of states. Our Theorem 1 is used to solve for the social planner's optimal information revelation scheme. Below, we provide two additional applications of our methods.

6.1 Motivating through strategic disclosure

A principal wants to motivate an agent to exert costly effort in order to complete a project.¹¹ The project, if completed successfully, has value X to the principal. We assume that the agent receives a fraction $\beta \in (0, 1)$ of the value of the project. The agent chooses effort level $e \in [0, \bar{e}]$, with $\bar{e} \leq 1$, where e is interpreted as probability that the project will be successfully completed. Choosing effort level e has disutility $c(e) = e^\alpha$, where $\alpha > 1$.¹²

The value X is distributed according to a continuous full-support distribution F on $[\underline{x}, \bar{x}]$. The principal observes the realization x of X , but the agent does not. The principal commits to a disclosure scheme that maximizes her expected profits.

Given the belief of the agent that the expected value of the project is y , the chosen level of effort is equal to

$$e^*(y) = \min \left\{ \left(\frac{\beta y}{\alpha} \right)^{\frac{1}{\alpha-1}}, \bar{e} \right\}.$$

Let $\bar{y} = (\alpha/\beta)\bar{e}^{\alpha-1}$ be the smallest expected value of the project such that $e^*(\bar{y}) = \bar{e}$, i.e. the agent exerts maximal effort. To make the analysis interesting, we assume that $\underline{x} < \bar{y} < \bar{x}$.

The principal's utility function is $u(y) = (1 - \beta)e^*(y)y$. It is easy to check that u is strictly convex on $[\underline{x}, \bar{y}]$, and affine on $[\bar{y}, \bar{x}]$. The shape of the function $u(\cdot)$ is depicted in Figure 7.

Proposition 3. *If $\mathbb{E}X \geq \bar{y}$, it is optimal to reveal no information. Otherwise, let x^* be defined by $\mathbb{E}[X | X \geq x^*] = \bar{y}$. Then, it is optimal to disclose x whenever $x < x^*$, and to reveal only that*

¹¹A similar application (with binary state space) is studied in Section 6.2 of [Kamenica and Gentzkow \(2009\)](#).

¹²All the results in this section continue to hold under the assumption that $(c')^{-1}(e) \cdot e$ is convex, without any particular choice of functional form.

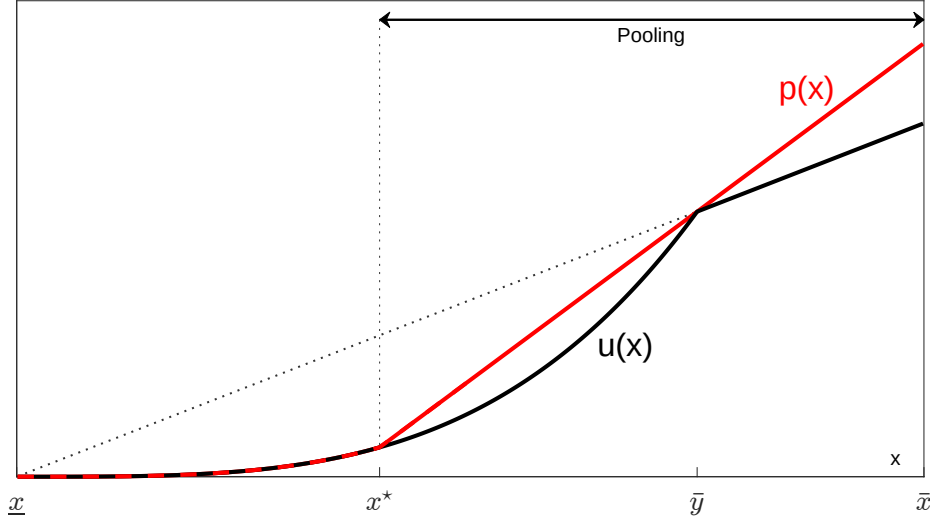


Figure 7: Motivating through strategic disclosure

$x \geq x^*$ if $x \geq x^*$.

Proof. In the first case, by the second paragraph of Section 4, it is optimal not to reveal anything.

In the second case, consider the price function

$$p(x) = \begin{cases} u(x) & \text{if } x < x^* \\ u(x^*) + \Delta(x - x^*) & \text{if } x \geq x^* \end{cases}, \quad \text{where } \Delta \equiv \frac{u(\bar{y}) - u(x^*)}{\bar{y} - x^*}.$$

Function p is convex. We then define the cdf G of posterior means as

$$G(x) = \begin{cases} F(x) & \text{if } x < x^* \\ F(x^*) + \mathbf{1}_{\{x \geq \bar{y}\}}(1 - F(x^*)) & \text{if } x \geq x^* \end{cases}.$$

That is, G coincides with F up to x^* , and then puts an atom at $\bar{y} = \mathbb{E}[X | X \geq x^*]$. Condition (3.1) holds because u and p coincide for $x \leq x^*$ and $x = \bar{y}$. Condition (3.2) is satisfied because p is affine whenever $F \neq G$, and the conditional means of F and G are equal in that region. By the way we defined G , F is a mean-preserving spread of G , which verifies condition (3.3). By Theorem 1, G is optimal. $\square$

Proposition 3 has the following economic interpretation. If the agent believes the value of the project to be high enough ex-ante ($\mathbb{E}X \geq \bar{y}$), he exerts maximal effort, so it is optimal for the principal to release no additional information. In the opposite case $\mathbb{E}X < \bar{y}$, the principal uses an “upper censorship” rule. She discloses the exact value of the project for low realizations but garbles the signal for high realizations by only informing the agent that the project deserves maximum effort.

6.2 Investment recommendation

A risk-averse investor chooses how to divide her wealth w between a risk-free asset and a single risky asset, on which she can take a short or a long position. (A similar analysis is possible if only one of the two positions is available.) The amount invested in the risky asset either doubles in value or is entirely lost. To take any non-zero position in the risky asset, the investor must pay a fixed cost $c > 0$. The investor has prior belief $\frac{1}{2}$ that a long position will double her investment.

The investor consults a financial analyst, who has access to additional information about the payoff of the risky asset. Specifically, she knows the probability X that a long position will double the investment. (With probability $1 - X$ a short position will double the investment.) X is distributed according to F which is symmetric around the mean $\mathbb{E}[X] = \frac{1}{2}$ and admits a strictly positive density on $[0, 1]$. The analyst commits to a persuasion mechanism in order to influence the belief of the agent about the payoff of the risky asset. If the posterior belief is too close to the prior, because of the fixed cost the agent will optimally invest zero. The closer to 0 or 1 the belief, the more the agent is willing to invest.

We consider two different shapes of the analyst's utility u . In the first (Figure 8), as posterior belief approaches 0 and 1, the function is concave, meaning there are diminishing returns from inducing polarized beliefs. In the second (Figure 9), the function u is convex near 0 and 1. In both cases u is flat near $\frac{1}{2}$, where there is no investment. We provide a microfoundation of these shapes in Appendix A.6 assuming that the investor has CRRA preferences and the analyst's utility is proportional to the amount invested. The shape of u depends on the measure of relative risk aversion, with u becoming more convex as risk aversion increases. The linear boundary case is attained when the investor has log-utility. Similar shapes can also arise because of non-linearities in the analyst's fee structure.

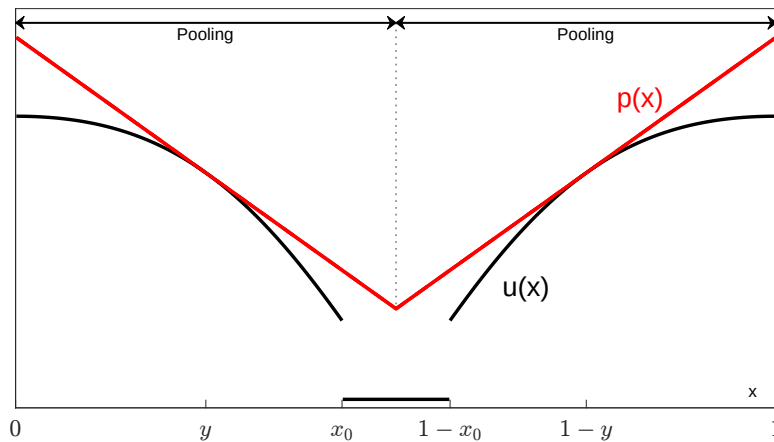


Figure 8: Portfolio recommendation: concave case

In Proposition 4 we solve the analyst's persuasion problem under the assumption that the analyst is, on average, sufficiently informed to always induce positive investment in the risky asset,

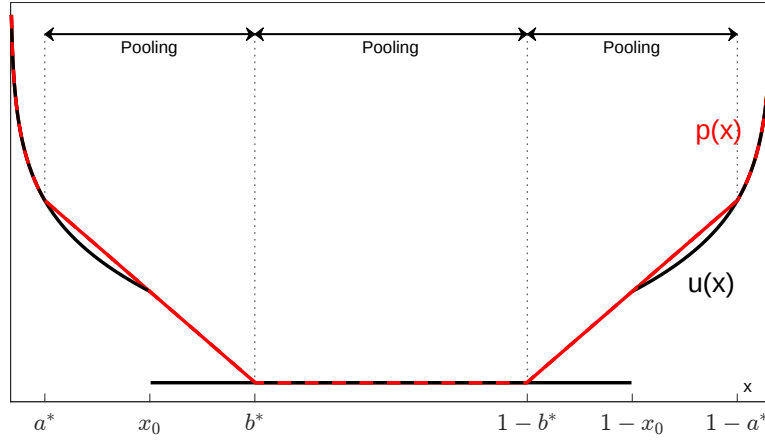


Figure 9: Portfolio recommendation: convex case

when information is pooled in two signals (high and low). See Appendix A.6 for a formal statement of this assumption, for the parametrization of u , and for the proof of Proposition 4.

Proposition 4. *In the concave case (Figure 8), it is optimal to reveal whether $x < \frac{1}{2}$ or $x > \frac{1}{2}$. In the convex case (Figure 9), there exist $a^* < b^*$ such that it is optimal to disclose x whenever $x \leq a^*$ or $x \geq 1 - a^*$, and to pool realizations (i) $x \in (a^*, b^*)$ at $x_0 = E[X|a^* < X < b^*]$; (ii) $x \in (1 - b^*, 1 - a^*)$ at $1 - x_0$; (iii) $x \in [b^*, 1 - b^*]$ at $\frac{1}{2}$.*

Proposition 4 can be interpreted as follows. If the agent is not highly risk averse (with the CRRA microfoundation, if the agent is less risk averse than log-utility; see Appendix A.6 for details), the analyst issues a binary recommendation “buy” or “sell”, and the investor always invests a positive amount of wealth in the position recommended by the analyst. If the agent is more risk averse, the recommendation of the analyst has a finer structure. If the analyst thinks the asset will go up (or down) with high probability, she provides full information about the assessed probability. If she is less confident, she issues a “weak recommendation” to either buy or sell. And if the realized x is close to the prior belief $\frac{1}{2}$, she does not recommend any of the positions (“hold”). In this last case, the agent refrains from investing.

7 Extensions

7.1 Competition in persuasion

In this subsection, we show that our methods extend to a setting where there are multiple Senders that have access to the same information. We adopt the competition in persuasion model of Gentzkow and Kamenica (2016) specialized to our setting. Formally, there are N Senders, and each Sender i maximizes the expectation of some function $u_i(x)$, where x is the mean of the posterior distribution (given the information revealed by all Senders). Senders choose the distributions of

signals simultaneously, and can send arbitrarily correlated signals. We study Nash equilibria of this game.

By the results of [Gentzkow and Kamenica \(2016\)](#), a distribution H over posterior means is an equilibrium distribution of joint information revealed by all Senders if and only if it is *unimprovable* – no Sender i wants to reveal more information given H .¹³ Therefore, to characterize all equilibrium distributions, we only need to understand which distributions are unimprovable. We say that $v : [0, 1] \rightarrow \mathbb{R}$ is a *convex translation* of $u : [0, 1] \rightarrow \mathbb{R}$ if $v - u$ is convex. Because our next result pertains to any Sender i , we drop the subscript.

Theorem 4. *Suppose that u is continuous and regular. Then, H is unimprovable for a Sender with utility function u if and only if there exists a function $\hat{u}$ that is a convex translation of u , and a convex function $p : [0, 1] \rightarrow \mathbb{R}$, $p \geq \hat{u}$, such that (H, p) satisfy conditions (3.1)–(3.3) with u replaced by $\hat{u}$. (The “if” part requires no additional assumptions on u .)*

We sketch the proof in [Appendix A.7](#).

Theorem 4 leads to an economically meaningful algorithm for generating the set of equilibrium distributions for the competition in persuasion model: (1) For all $i \in N$, solve the individual optimization problem, using the tools developed in preceding sections, for all Senders that are more risk-loving (in the order induced by convex translations) than Sender i ; denote the set of obtained solutions by $\mathcal{H}_i$; (2) The set of equilibrium distributions is the intersection $\bigcap_{i \in N} \mathcal{H}_i$.

We show how to apply the above method to solve a multi-Sender version of the application in [Section 6.1](#). Suppose that there are $N \geq 2$ Senders that observe X and compete to persuade the agent. We assume that utility u_i of each Sender i is: (1) strictly convex on $[0, c_i]$ for some $c_i \in (0, 1)$; (2) affine on $[c_i, 1]$; (3) continuous, regular, and not globally convex. That is, each u_i takes the shape analogous to the shape of u in [Figure 7](#), but the locations c_i of the kink may be different across Senders. Without loss of generality, we can order the Senders so that $c_1 \geq c_2 \geq \dots \geq c_N$. To limit the number of cases, assume that the prior mean satisfies $\mathbb{E}_F[X] < c_N$.

Proposition 5. *H is an equilibrium distribution if and only if H is feasible, fully reveals the state below some $a \leq c_1$, and has no mass on (a, c_1) . In particular, the least informative equilibrium is the distribution that would be optimal for Sender 1 if she were the only Sender.*

Proof. See [Appendix A.8](#). □

7.2 Multidimensional persuasion

Many of the tools developed so far (the existence guarantee of [Theorem 2](#); the characterization of prices of [Proposition 2](#)) rely on the properties of mean-preserving spreads that hold only in the univariate case. However, the logic of [Theorem 1](#) extends immediately to the multivariate setting where the Sender’s utility only depends on the vector of posterior means.

¹³Formally, H is unimprovable for Sender i if $G = H$ maximizes $\int_0^1 u_i(x) dG(x)$ over G subject to F being a MPS of G and G being a MPS of H .

The state of nature is now the realization of a random vector X with distribution F over the unit hypercube $Q := [0, 1]^k$, for some fixed $k \geq 2$. Mean-preserving spreads generalize to the convex order among distributions: $G \leq_{cx} F$ if and only if, for all convex v , $\int v dG \leq \int v dF$. Unlike the mean-preserving spread relation, there is no convenient characterization of the convex order in dimensions 2 or greater. To verify that it holds for some G , one can use a suitable version of Strassen's theorem (e.g. Theorem 3.4.2(a) in Müller and Stoyan, 2002): If G is the distribution of a random variable Y that satisfies $Y = E[X|Y]$, then $G \leq_{cx} F$.

Theorem 1b. *If G is a distribution on Q , $p : Q \rightarrow \mathbb{R}$ is convex with $p(x) \geq u(x)$, $\forall x \in Q$, and (G, p) satisfy*

$$\text{supp}(G) \subseteq \{x \in Q : u(x) = p(x)\}, \quad (7.1)$$

$$\int_Q p(x) dF(x) = \int_Q p(x) dG(x), \quad (7.2)$$

$$F \geq_{cx} G, \quad (7.3)$$

then G is optimal for the Sender's problem $\max_{G: G \leq_{cx} F} \int_Q u(x) dG(x)$.

The proof is identical to that of Theorem 1 and thus omitted. We apply Theorem 1b to a two-dimensional persuasion problem next.

The Sender wants to persuade the Receiver to take action on two dimensions, labeled 1 and 2. Receiver's payoff (v_1, v_2) from taking actions 1 and 2, respectively, is observed by the Sender but not the Receiver. We assume that actions are costly for the agent, so that the agent will take action j if and only if its expected utility v_j is at least β , for some fixed $\beta \in (0, 1)$.

The Sender has state-independent preferences and on both dimensions prefers the Receiver to take action. The Sender's utility is 0 if the agent takes neither action; 1 if the agent takes one action; and $2 + \alpha$ if the agent takes both actions. The parameter of interest is the degree of complementarity α . If $\alpha > 0$, actions are complements from the point of view of the Sender; if $\alpha \in (-1, 0)$, they are substitutes.¹⁴

We make the following assumptions, both for technical reasons and to isolate the most interesting case, in which β is relatively high (and thus persuasion is difficult). First, assume that (v_1, v_2) is distributed according to a symmetric distribution F with a density f which is strictly positive and bounded on $[0, 1]^2$. Second, assume that for all the following regions, the x component of their center of mass according to F is less than or equal to β : (1) $\{(x, y) : x \geq y, x + y \leq \underline{s}\}$, for all $\underline{s} \leq 2\beta$; (2) $\{(x, y) : x + y \geq \beta\}$; (3) $\{(x, y) : x + y \geq \beta, x \leq \beta, y \leq \beta\} \cup \{(x, y) : x + y \geq 2\beta\}$. For any fixed F , for large enough β the assumption is satisfied. If F is the bivariate uniform distribution, the assumptions hold if $\beta \geq 2/3$.

¹⁴The case in which it is the Receiver who has complementary or substitutable preferences can also be studied in this framework, by fixing the Sender's utility levels and instead varying the region where both actions are chosen by the Receiver.

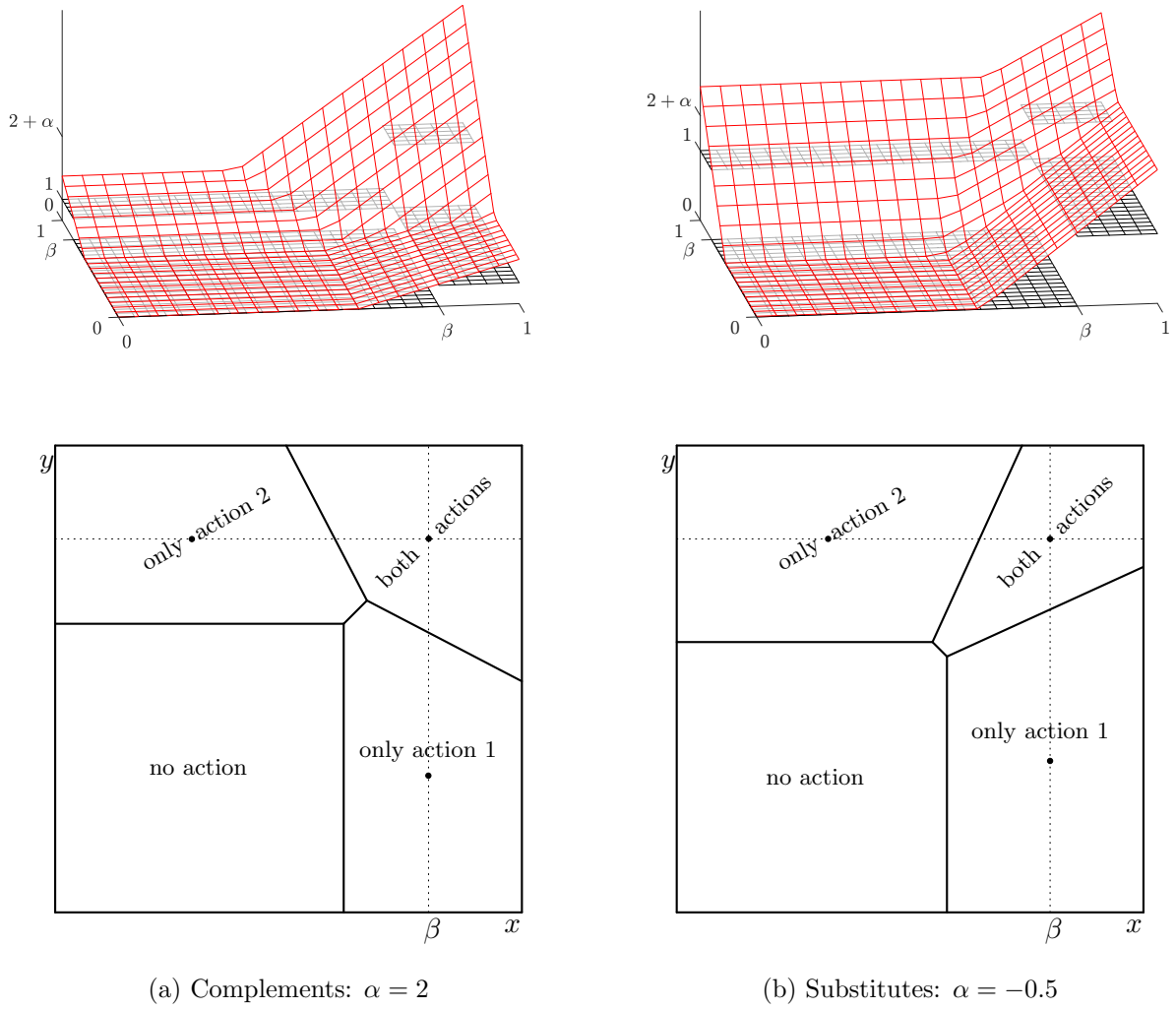


Figure 10: Solution of the multidimensional persuasion problem for complements (left panels) and substitutes (right panels). The top row shows u (fine mesh, in background) and p (coarse mesh, in foreground). The bottom row shows the pooling regions, which actions they induce, and how the centers of mass of the regions that induce some action lie on the threshold β . The pooling regions correspond to the affine regions of p . Other parameters: $\beta = 0.8$, $F =$ bivariate uniform.

Proposition 6. *For all $\alpha > -1$, there exist a price function p and a distribution G that satisfy the conditions of Theorem 1b. The function p is the pointwise maximum of four affine functions; the optimal G is obtained by pooling realizations of X in each of the regions on which p is affine.*

Proof. See Appendix A.9. □

The Sender faces a non-trivial trade-off between increasing the probability of inducing both actions versus the probability of inducing at least one action. Only in the limit as $\alpha \rightarrow \infty$ does the Sender maximize the probability of inducing both actions (which is achieved by pooling all states (x, y) with $x + y \geq v$, for an appropriately chosen v).

8 Prices in the general Bayesian persuasion problem

In this section, we provide an analog of Theorem 1 in the general Bayesian persuasion problem (Kamenica and Gentzkow, 2011), where the Sender's utility depends on the distribution of posterior beliefs, and not necessarily on posterior means only. We then illustrate the connection between the concavification approach and our approach based on prices. We conclude by solving an example.

Let Θ be a compact metric space, and suppose that the Sender's utility $V : \Delta(\Theta) \rightarrow \mathbb{R}$ is upper semi-continuous. Let $\mu_0 \in \Delta(\Theta)$ be the prior distribution. The general Bayesian persuasion problem is

$$\max_{\tau \in \Delta(\Delta(\Theta))} \int_{\Delta(\Theta)} V(\mu) d\tau(\mu) \quad (8.1)$$

subject to the Bayes-plausibility constraint

$$\int_{\Delta(\Theta)} \mu d\tau(\mu) = \mu_0. \quad (8.2)$$

Before stating the result, we introduce a class of functions that we call “outer-convex.”

Definition 3. A function $P : \Delta(\Theta) \rightarrow \mathbb{R}$ is *outer-convex* if, for any $\mu \in \Delta(\Theta)$,

$$P(\mu) \leq \int_{\Theta} P(\delta_{\theta}) d\mu(\theta),$$

where δ_{θ} is a measure that puts all mass on $\{\theta\}$ (Dirac delta).

Outer-convexity relaxes convexity by only requiring the relevant inequality to hold for convex combinations supported on extreme points of the domain. When the domain is $\Delta(\Theta)$, extreme points are exactly Dirac deltas, and the definition of outer-convexity has a natural interpretation in the Bayesian persuasion framework: Outer-convexity of V is equivalent to optimality of full disclosure for any prior.

Theorem 5. Suppose that there exists $\tau^* \in \Delta(\Delta(\Theta))$ and a measurable outer-convex function $P : \Delta(\Theta) \rightarrow \mathbb{R}$, such that $P \geq V$, and

$$\text{supp}(\tau^*) \subseteq \{\mu \in \Delta(\Theta) : V(\mu) = P(\mu)\}, \quad (8.3)$$

$$\int_{\Delta(\Theta)} P(\mu) d\tau^*(\mu) = \int_{\Theta} P(\delta_{\theta}) d\mu_0(\theta), \quad (8.4)$$

$$\int_{\Delta(\Theta)} \mu d\tau^*(\mu) = \mu_0, \quad (8.5)$$

Then, τ^* is a solution to problem (8.1) – (8.2).

Proof. Identical to the proof of Theorem 1 (omitted). □

Theorem 5 is analogous to Theorem 1 except that the outer-convex price functional P is defined on beliefs rather than on posterior means – this is natural given that in the general Bayesian persuasion problem the preferences of the Sender depend on the entire belief profile.

In the competitive equilibrium interpretation, the consumer is endowed with the prior μ_0 whose market value under prices P is given by $\int_{\Theta} P(\delta_{\theta}) d\mu_0(\theta)$. The consumer purchases posterior beliefs μ at prices $P(\mu)$ to maximize her quasi-linear utility – condition (8.3) can be equivalently written as $\text{supp}(\tau^*) \subseteq \text{argmax}_{\mu \in \Delta(\Theta)} \{V(\mu) - P(\mu)\}$. The firm’s problem requires a modification in the general model. While the mean-preserving spread condition encodes the idea that the firm can “merge” states into posterior means, the Bayes-plausibility condition (8.2) by itself allows for both “merging” and “splitting” of posterior beliefs. Thus, the production possibility set of the firm must now include the Bayes-plausibility constraint but also a condition that only Dirac deltas can be used as inputs in the production process. With this modification, outer-convexity of prices is necessary for existence of a Walrasian equilibrium (when prices fail to be outer-convex at μ , the firm can make infinite profits by producing (and selling) infinite amounts of μ from its decomposition into Dirac deltas). Moreover, the firm breaks even (condition 8.4) and the markets clear (condition 8.5).

When the function V is (outer-)convex, the supporting prices are given by $P(\mu) = V(\mu)$. P is (outer) convex, $P \geq V$, and the three conditions of Theorem 5 follow immediately with τ^* corresponding to full disclosure. When the function V is concave, let $\Phi_{\mu_0}(\mu)$ be the (affine) function whose graph is the supporting hyperplane of V at μ_0 . With $P(\mu) = \Phi_{\mu_0}(\mu)$, $P \geq V$, and conditions (8.3) – (8.5) hold trivially when $\tau^*(\mu_0) = 1$, i.e., τ^* corresponds to no disclosure. Finally, suppose that the preferences of the Sender depend only on the posterior mean, $V(\mu) = u(\mathbb{E}_{\mu}(\theta))$, and let (G, p) satisfy conditions (3.1)–(3.3). Then, take $P(\mu) = p(\mathbb{E}_{\mu}(\theta))$. Conditions (8.3) – (8.5) with τ^* corresponding to G follow from the corresponding conditions of Theorem 1.

Theorem 5 allows us to draw a connection between our approach and the concavification approach of Kamenica and Gentzkow (2011).

Claim 1. *Suppose that P^* is the outer-convex price functional satisfying the conditions of Theorem 5 (under the prior μ_0 and with some optimal distribution τ^*). Then*

$$\text{co}(V)(\mu_0) = \min \left\{ \int_{\Theta} P(\delta_{\theta}) d\mu_0(\theta) \mid P : \Delta(\Theta) \rightarrow \mathbb{R}, P \text{ is outer-convex}, P \geq V \right\}. \quad (8.6)$$

Moreover, the minimum is attained at $P = P^*$:

$$\text{co}(V)(\mu_0) = \int_{\Theta} P^*(\delta_{\theta}) d\mu_0(\theta). \quad (8.7)$$

Proof. See Appendix A.10. □

Claim 1 shows the dual relationship between the concave closure of V (which is the value function for the problem (8.1) – (8.2)) and the (outer-)convex price functional P from Theorem 5. The value of the persuasion problem is equal to the value of the endowment at the equilibrium prices. Equation (8.7) can be interpreted in the following way. If V is replaced with its concave

closure $\text{co}(V)$ (which is pointwise greater) as the objective function of the Sender, it is optimal to reveal *nothing* and the value of the persuasion problem stays the same. Analogously, if V is replaced with P^* (which is pointwise greater) as the objective function of the Sender, it is optimal to reveal *everything* and the value of the persuasion problem stays the same.¹⁵ In this sense, P^* can be seen as a (prior-dependent) “convex closure” of V .

Theorem 5 can be used to solve problems in which the designer’s objective function depends on more than one moment of the posterior distribution. We solve one such example next.

8.1 Example

We consider an extension of the example from Section 6.1. There, we assumed that the agent had utility which was linear in money. However, it is natural to study a case when the agent’s utility exhibits decreasing marginal returns. Formally, in the setting of Section 6.1, suppose that the agent evaluates monetary payoffs βx according to a concave utility function $u_A(\beta x)$, where x is the realization of the value of the project X , and β is the share received by the agent. To simplify calculations, we assume that the cost of effort is quadratic, so that the problem of the agent becomes

$$\max_{e \in [0, \bar{e}]} \{e \mathbb{E} u_A(\beta X) - e^2\}.$$

The optimal effort level is thus a function of $y = \mathbb{E} u_A(\beta X)$:

$$e^*(y) = \min \left\{ \bar{e}, \frac{y}{2} \right\}.$$

This means that the utility of the risk-neutral Sender will now depend on two moments of the posterior distribution: its mean and its expected utility for the agent. Formally, the principal maximizes

$$V(\mu) = (1 - \beta) e^*(\mathbb{E}_\mu u_A(\beta X)) \mathbb{E}_\mu X,$$

and the objective function depends on the posterior distribution μ through $y = \mathbb{E}_\mu u_A(\beta X)$ and $x = \mathbb{E}_\mu X$. Let $v(x, y) = (1 - \beta) e^*(y) x$.

Let F be a cdf of a full-support distribution of X on $[\underline{x}, \bar{x}]$, and, to limit the number of cases to consider, suppose that $e^*(\mathbb{E}_F u_A(\beta X)) < \bar{e}$ (the agent take less than full effort under the prior), and that $e^*(u_A(\beta \bar{x})) = \bar{e}$ (the agent takes full effort under the most optimistic belief about X).

Proposition 7. *Suppose that $u_A(x)x$ is convex. Let x^* be the lowest x such that the prior belief truncated at x induces full effort: $\mathbb{E}_F[u_A(\beta X) | X \geq x^*] = 2\bar{e}$. Then, it is optimal to disclose x whenever $x < x^*$, and to reveal only that $x \geq x^*$ if $x \geq x^*$.*

Proof. See Appendix A.11. □

The optimal persuasion scheme from Proposition 7 is the same as that from Proposition 3 except that x^* is defined with respect to the concave utility function u_A of the agent. While the solution

¹⁵The last claim follows from conditions (8.3) - (8.4).

is intuitive, proving its optimality is challenging. We do so by applying Theorem 5. Because the objective function depends on two moments of the posterior belief, it is natural to conjecture that so does the price function. We construct a two-dimensional function $p(x, y)$ that majorizes $v(x, y)$, and show that $P(\mu) = p(\mathbb{E}_\mu X, \mathbb{E}_\mu u_A(\beta X))$ together with the disclosure policy described in Proposition 7 satisfy the conditions of Theorem 5. The assumption that $u_A(x)x$ is convex allows us to prove that V restricted to the region where interior effort is optimal for the agent is outer-convex, an analog of convexity of the one-dimensional objective function $u(x)$ for $x \leq \bar{y}$ in Figure 7. Figure 11 depicts the objective function and the supporting prices in the two-dimensional space $(x, y) = (\mathbb{E}_\mu X, \mathbb{E}_\mu u_A(\beta X))$. The objective function has a non-differentiable edge described by the equation $y = \bar{y} \equiv 2\bar{e}$ corresponding to the level of utility that is just enough to induce full effort. The price function p is defined as a pointwise maximum of v and a hyperplane “tangent” to the graph of v along this non-differentiable edge.

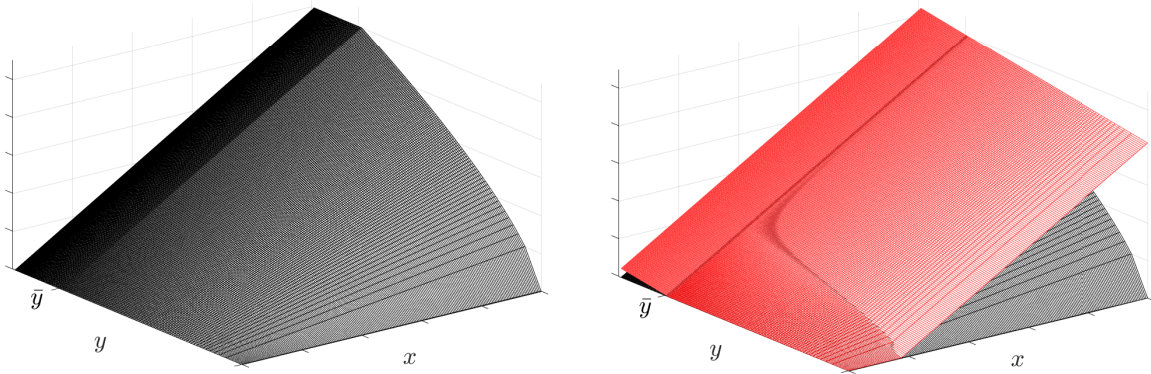


Figure 11: The objective function $v(x, y)$ (left panel) and the supporting prices $p(x, y)$ (right panel)

The above example suggests that, more generally, if the objective function is measurable with respect to a set of moments of the posterior distribution, so is the corresponding price functional. We leave this direction for future research.

A Appendix

A.1 Proof of Theorem 1¹⁶

Let (G, p) satisfy conditions (3.1)–(3.3). To show that G is a solution to the Sender’s problem, it is enough to show that $\int_0^1 u(x) dG(x) \geq \int_0^1 u(x) dH(x)$, for any H such that F is a mean-preserving spread of H .

¹⁶We thank Shota Ichihashi for showing us this direct proof. Our previous proof employed duality techniques from the literature on optimization with stochastic dominance constraints, which we use in the proof of Theorem 2.

By (3.1), and the fact that $p \geq u$,

$$\int_0^1 (u(x) - p(x))dG(x) \geq \int_0^1 (u(x) - p(x))dH(x).$$

Rearranging,

$$\begin{aligned} \int_0^1 u(x)dG(x) - \int_0^1 u(x)dH(x) &\geq \int_0^1 p(x)dG(x) - \int_0^1 p(x)dH(x) \\ &\stackrel{(1)}{=} \int_0^1 p(x)dF(x) - \int_0^1 p(x)dH(x) \\ &\stackrel{(2)}{\geq} \int_0^1 p(x)dF(x) - \int_0^1 p(x)dF(x) = 0 \end{aligned}$$

where equality (1) follows from (3.2) and inequality (2) holds because $-p(x)$ is concave, and F is a mean-preserving spread of H , by assumption. Therefore $\int_0^1 u(x)dG(x) \geq \int_0^1 u(x)dH(x)$. $\square$

A.2 Proof of Theorem 2

We prove the theorem in three steps. In the first step, we make the additional assumption that u is continuous, and study a perturbed problem. The perturbation allows us to show that a generalized Slater condition holds, which results in existence of the appropriate Lagrange multiplier. In the second step, we show that the solution to the perturbed problem provides the correct approximation of the solution to the unperturbed problem. By taking the limit, we obtain the statement of Theorem 2 for the special case of a continuous u . In the third step, we relax the assumption that u is continuous, again using an approximation approach. We relegate some technical lemmas to Appendix A.12.

A solution to the Sender's problem always exists because the objective function is upper semi-continuous, and the set of feasible points is compact in the weak* topology. By Fact 1, to show that a supporting price function p exists for all optimal solutions, it is enough to prove that it exists for one solution. Fact 1 follows from the following string of inequalities:

$$\begin{aligned} \int_0^1 p(x)dF(x) &= \int_0^1 p(x)dG(x) = \int_0^1 u(x)dG(x) = \int_0^1 u(x)dG'(x) \\ &\leq \int_0^1 p(x)dG'(x) \leq \int_0^1 p(x)dF(x). \end{aligned}$$

Thus, we have equalities everywhere, and hence (G', p) satisfy (3.1) and (3.2). Condition (3.3) holds because G' is feasible.

Step 1. In the first step, we use the proof technique developed by [Dentcheva and Ruszczyński \(2003\)](#). [Dentcheva and Ruszczyński](#) provide a duality theory for optimization problems with stochastic dominance constraints. They study a case where the constraint takes the form of second-

order stochastic dominance. Our constraint additionally incorporates equality of unconditional means, resulting in a mean-preserving spread condition.

Assume that u is continuous. Consider the following perturbed problem, where the mean-preserving spread condition is only imposed on the interval $[\epsilon, 1 - \epsilon]$, instead of on $[0, 1]$.

$$\max_{G \in \Delta([0, 1])} \int_0^1 u(x) dG(x) \quad (\text{A.1})$$

$$\text{s.t. } \int_0^x F(t) dt \geq \int_0^x G(t) dt \text{ for all } x \in [\epsilon, 1 - \epsilon], \quad (\text{A.2})$$

$$\int_x^1 G(t) dt \geq \int_x^1 F(t) dt \text{ for all } x \in [\epsilon, 1 - \epsilon]. \quad (\text{A.3})$$

Conditions (A.2) and (A.3) enforced on the entire interval $[0, 1]$ would be jointly equivalent to F being a mean-preserving spread of G . Note that although the perturbed problem (A.1) only requires the mean-preserving spread condition on $[\epsilon, 1 - \epsilon]$, all distributions are defined on $[0, 1]$. We take $\epsilon > 0$ small enough so that $\epsilon < \mathbb{E}[F] < 1 - \epsilon$.

Let $\mathcal{C}_\epsilon := \mathcal{C}([\epsilon, 1 - \epsilon])$ denote the space of continuous functions on $[\epsilon, 1 - \epsilon]$. Define K_ϵ as the cone of continuous non-negative functions on $[\epsilon, 1 - \epsilon]$, that is,

$$K_\epsilon := \{g \in \mathcal{C}_\epsilon : g(x) \geq 0, \forall x \in [\epsilon, 1 - \epsilon]\}.$$

Let $\Phi : \Delta([0, 1]) \rightarrow \mathcal{C}_\epsilon \times \mathcal{C}_\epsilon$ be defined by

$$(\Phi(G))_i(x) = \begin{cases} \int_0^x F(t) dt - \int_0^x G(t) dt & \text{if } i = 1, \\ \int_x^1 G(t) dt - \int_x^1 F(t) dt & \text{if } i = 2, \end{cases}$$

for any $x \in [\epsilon, 1 - \epsilon]$. Conditions (A.2) and (A.3) are now equivalent to $\Phi(G) \in K_\epsilon \times K_\epsilon$.

The operator Φ is concave with respect to the product cone $K_\epsilon \times K_\epsilon$. By the Riesz representation theorem, the space dual to $\mathcal{C}_\epsilon$ is the space $\mathbf{rca}([\epsilon, 1 - \epsilon])$ of regular countably additive measures on $[\epsilon, 1 - \epsilon]$ having finite variation. We define a Lagrangian $\Lambda : \Delta([0, 1]) \times \mathbf{rca}([\epsilon, 1 - \epsilon]) \times \mathbf{rca}([\epsilon, 1 - \epsilon]) \rightarrow \mathbb{R}$,

$$\Lambda(G, \mu_1, \mu_2) = \int_0^1 u(x) dG(x) + \int_\epsilon^{1-\epsilon} (\Phi(G))_1(x) d\mu_1(x) + \int_\epsilon^{1-\epsilon} (\Phi(G))_2(x) d\mu_2(x).$$

We now show that the generalized Slater condition holds for the problem (A.1). By [Bonnans and Shapiro \(2000\)](#) (Proposition 2.106), it is enough to show that there exists $\tilde{G} \in \Delta([0, 1])$ such that

$$\Phi(\tilde{G}) \in \text{int}(K_\epsilon \times K_\epsilon). \quad (\text{A.4})$$

That is, we have to find a distribution (cdf) $\tilde{G}$ supported on $[0, 1]$ such that $\Phi(\tilde{G})$ is a Cartesian product of two functions that are both in K_ϵ , and are bounded away from zero on $[\epsilon, 1 - \epsilon]$. Let

$\tilde{G}(x) = \mathbf{1}_{\{x \geq \mathbb{E}[F]\}}$. The function $(\Phi(\tilde{G}))_i(x)$, for $i = 1, 2$, is equal to 0 at $x = 0$ and $x = 1$. Since 0 and 1 are in the support of F , $(\Phi(\tilde{G}))_i(x)$ is strictly positive in the interior of $[0, 1]$. Therefore, using the structure of $\tilde{G}$, it must be bounded away from zero on $[\epsilon, 1 - \epsilon]$. Thus, the generalized Slater condition holds.

By [Bonnans and Shapiro \(2000\)](#) (Theorem 3.4), we conclude that if G is a solution to the problem (A.1) (a solution exists by the comment from the beginning of the proof), then there exist non-negative measures $\mu_1^*, \mu_2^* \in \mathbf{rca}([\epsilon, 1 - \epsilon])$ such that

$$\Lambda(G, \mu_1^*, \mu_2^*) = \max_{\hat{G} \in \Delta([0, 1])} \Lambda(\hat{G}, \mu_1^*, \mu_2^*), \quad (\text{A.5})$$

and

$$\int_{\epsilon}^{1-\epsilon} (\Phi(G))_i(x) d\mu_i^*(x) = 0, \quad i = 1, 2. \quad (\text{A.6})$$

Using an argument analogous to the one used by [Dentcheva and Ruszczyński \(2003\)](#), we can associate with each measure μ_i^* a function $p_i^* : [0, 1] \rightarrow \mathbb{R}$,

$$p_1^*(x) = \begin{cases} \int_x^{1-\epsilon} \mu_1^*([\tau, 1 - \epsilon]) d\tau, & x < 1 - \epsilon \\ 0, & x \geq 1 - \epsilon, \end{cases}$$

$$p_2^*(x) = \begin{cases} 0, & x < \epsilon \\ \int_{\epsilon}^x \mu_2^*([\epsilon, \tau]) d\tau, & x \geq \epsilon, \end{cases}$$

where each μ_i^* is extended to $[0, 1]$ by putting zero mass beyond the interval $[\epsilon, 1 - \epsilon]$. By the properties of μ_i^* , the function p_1^* is non-increasing and convex, and p_2^* is non-decreasing and convex.

We have (by changing the order of integration)

$$\int_{\epsilon}^{1-\epsilon} \left(\int_0^x G(t) dt \right) d\mu_1^*(x) = \int_0^1 \left(\int_{\epsilon}^{1-\epsilon} \mathbf{1}_{\{t \leq x\}} d\mu_1^*(x) \right) G(t) dt = \int_0^1 \mu_1^*([t, 1 - \epsilon]) G(t) dt.$$

Using the definition of p_i^* , we can write

$$\int_{\epsilon}^{1-\epsilon} \left(\int_0^x G(t) dt \right) d\mu_1^*(x) = - \int_0^1 G(t) dp_1^*(t).$$

Similarly, we have

$$\int_{\epsilon}^{1-\epsilon} \left(\int_x^1 G(t) dt \right) d\mu_2^*(x) = \int_0^1 G(t) dp_2^*(t).$$

Using integration by parts, we get

$$\int_0^1 G(t) dp_i^*(t) = p_i^*(1) - \int_0^1 p_i^*(x) dG(x).$$

Therefore, the complementary-slackness condition (A.6) becomes

$$\int_0^1 p_i^*(x) dG(x) = \int_0^1 p_i^*(x) dF(x), \quad i = 1, 2.$$

Finally, define $p(x) = p_1^*(x) + p_2^*(x)$. Then, $p(x)$ is convex, and obviously satisfies

$$\int_0^1 p(x) dG(x) = \int_0^1 p(x) dF(x). \quad (\text{A.7})$$

Finally, condition (A.5) implies that

$$G \in \operatorname{argmax}_G \left\{ \int_0^1 u(x) dG(x) - \int_0^1 p(x) dG(x) \right\}. \quad (\text{A.8})$$

We can always add a constant to p without changing any of its properties. Because of property (A.8), we can normalize p so that condition (3.1) holds. That is, $p \geq u$ and

$$\operatorname{supp}(G) \subseteq \{x \in [0, 1] : u(x) = p(x)\}. \quad (\text{A.9})$$

Therefore, for the perturbed problem, we have shown that there exists a convex $p : [0, 1] \rightarrow \mathbb{R}$ such that conditions (A.7) and (A.9) both hold.

Step 2. In this step, we show that we can take the limit of the perturbed problems from Step 1, and obtain a solution to the original (unperturbed) problem (still maintaining the assumption that u is continuous).

Consider a sequence of problems (A.1) - (A.3) defined by taking $\epsilon = 1/n$ for $n = 1, 2, \dots$. We obtain a sequence (G_n, p_n) of pairs satisfying conditions (A.7), (A.9), as well as (A.2) and (A.3).

We will first show a simplified version of the proof using the following assumption.

Assumption 1. The sequence of functions (p_n) converges uniformly on $[0, 1]$ to some convex function p .

In Appendix A.12, we present a full version of the proof without Assumption 1. This adds technical complications because the sequence p_n can potentially explode near the endpoints of the interval $[0, 1]$. In Appendix A.12, we show that we can still establish the result by appropriately modifying the functions p_n , and using the special structure of the problem.

Under Assumption 1, we have a well defined limit p of the sequence p_n . The sequence (G_n) , seen as a sequence of probability measures, lives in a compact set (in the weak* topology). Because the space of measures is metrizable, compactness is equivalent to sequential compactness, and thus we can choose a converging subsequence. Without loss of generality, G_n converges in the weak* topology to some distribution $G \in \Delta([0, 1])$.

We have thus defined the limiting pair (G, p) . We want to prove that (G, p) satisfies conditions (3.1)–(3.3) on $[0, 1]$.

First, as $n \rightarrow \infty$, conditions (A.2) and (A.3) imply that F is a mean-preserving spread of G on the entire interval $[0, 1]$. This establishes condition (3.3).

Second, we note that¹⁷

$$\text{supp}(G) \subseteq \limsup_n \{\text{supp}(G_n)\}.$$

Given that condition (A.9) holds for each n , and because p_n converges to p ,

$$\limsup_n \{\text{supp}(G_n)\} \subseteq \limsup_n \{x \in [0, 1] : u(x) = p_n(x)\} \subseteq \{x \in [0, 1] : u(x) = p(x)\}.$$

We conclude that

$$\text{supp}(G) \subseteq \{x \in [0, 1] : u(x) = p(x)\}, \quad (\text{A.10})$$

establishing condition (3.1).

Third, we will show that (3.2) holds. We will argue that $\int_0^1 p_n(x) dG_n(x) \rightarrow \int_0^1 p(x) dG(x)$. To see this, note that

$$\int_0^1 p_n dG_n - \int_0^1 p dG = \int_0^1 (p_n - p) dG_n - \int_0^1 p (dG - dG_n).$$

The second integral converges to zero by definition of convergence of G_n to G in the weak* topology. For the first integral, we have

$$\int_0^1 (p_n - p) dG_n \leq \sup_{x \in [0, 1]} \{|p_n(x) - p(x)|\}$$

which converges to zero because p_n converges to p uniformly on $[0, 1]$, by Assumption 1. Similarly, we have

$$\lim_n \int_0^1 p_n dF = \int_0^1 p dF.$$

Combining these two results with condition (A.7), we get

$$\int_0^1 p(x) dF(x) = \int_0^1 p(x) dG(x),$$

which is what we wanted to prove.

This finishes the proof of Step 2, i.e., we have shown that for a continuous u conditions (3.1)–(3.3) hold with (G, p) . By Theorem 1, G is the optimal solution.

Step 3. We now prove Theorem 2 without the additional assumption that u is continuous. As stated in the regularity assumption (Definition 1), u has finitely many one-sided jump discontinuities

¹⁷The lim sup of a sequence of sets A_n is defined as

$$\limsup_n A_n = \{x : \exists (x_n)_n \text{ s.t. } x_n \in A_n, \forall n, \text{ and } x_n \rightarrow x\}.$$

at $y_1, \dots, y_k \in (0, 1)$.

First, we construct a continuous approximation of u . Fix $\epsilon > 0$. We only modify the function u in an ϵ -neighborhood of each y_i . For small enough ϵ , these neighborhoods are disjoint. Take any $i = 1, \dots, k$, and suppose without loss of generality that $\lim_{y \uparrow y_i} u(y) < u(y_i) = \lim_{y \downarrow y_i} u(y)$. We denote by $\bar{u}_\epsilon$ the continuous function constructed by replacing u in each such neighborhood $[y_i - \epsilon, y_i]$ ¹⁸ with an affine majorant described above.

Because $\bar{u}_\epsilon$ is a continuous function with a bounded slope, by Steps 1 and 2, there exists an optimal distribution G_ϵ and a Lagrange multiplier p_ϵ such that (G_ϵ, p_ϵ) satisfies conditions (3.1)–(3.3).¹⁹

Lemma 1. *The sequence p_ϵ constructed by taking $\epsilon = 1/n$ and $n \rightarrow \infty$ has a subsequence that converges to some convex continuous p uniformly on $[0, 1]$.*

Proof. See Appendix A.12.

The lemma gives us a limit p of the sequence p_ϵ . By the same argument as in the first step of the proof, a subsequence of solutions G_ϵ as $\epsilon \rightarrow 0$ also converges to some distribution $G \in \Delta([0, 1])$.

The last step of the proof is to show that (G, p) satisfy conditions (3.1)–(3.3). This is immediate by the same arguments as used in the first step of the proof, and the fact that u and $\bar{u}_\epsilon$ coincide on the support of G_ϵ for small enough ϵ .²⁰

A.3 Proof of Proposition 1

To prove the first part, let G be an optimal solution to problem (2.1), and let p be the supporting price function (whose existence follows from Theorem 2). Let p^* be the solution to (3.7). We have

$$0 \stackrel{(1)}{\geq} \int_0^1 (u(x) - p^*(x)) dG(x) \stackrel{(2)}{=} \int_0^1 p(x) dG(x) - \int_0^1 p^*(x) dG(x) \quad (\text{A.11})$$

$$\stackrel{(3)}{=} \int_0^1 p(x) dF(x) - \int_0^1 p^*(x) dG(x) \stackrel{(4)}{\geq} \int_0^1 p(x) dF(x) - \int_0^1 p^*(x) dF(x) \stackrel{(5)}{\geq} 0, \quad (\text{A.12})$$

where (1) follows from the fact that $p^* \geq u$, (2) and (3) follow from the fact that (G, p) satisfy conditions (3.1)–(3.3), (4) follows from convexity of p^* and the fact that F is a MPS of G , and (5) follows from the fact that p^* solves (3.7). The above sequence of inequalities must be satisfied with equality which yields immediately that p^* is a supporting price function for G .

To prove the converse part, it is enough to use the same sequence of inequalities: If (G, p) satisfy (3.1)–(3.3), then all inequalities except for (5) hold for any feasible function $q = p^*$. We get that

$$\int_0^1 p(x) dF(x) - \int_0^1 q(x) dF(x) \leq 0$$

¹⁸Or $[y_i, y_i + \epsilon]$ if the function u “jumps down”, that is, its value is locally lower to the right of y_i .

¹⁹Note that we only need $\bar{u}_\epsilon$ to have a bounded slope for a fixed ϵ . We do not claim that the slope is bounded uniformly in ϵ .

²⁰Formally, this follows from the fact that p_ϵ have a uniformly bounded slope, as shown in the proof of Lemma 1.

for any feasible q , and thus p solves (3.7).

A.4 Proof of Proposition 2

Lemma 2. *If (G, p) satisfy conditions (3.2), (3.3), and p is convex and continuous, then*

$$\int_0^1 \left(\int_0^x F(t)dt - \int_0^x G(t)dt \right) dp'(x) = 0, \quad (\text{A.13})$$

interpreted as a Riemann-Stieltjes integral with respect to the measure induced by the non-decreasing function p' .

Proof. Because p is convex, it is absolutely continuous in the interior of the domain, and continuous at the endpoints by assumption. We can use integration by parts for the Riemann-Stieltjes integral:

$$\int_0^1 p(x)dG(x) = [p(x)G(x)]_0^1 - \int_0^1 G(x)dp(x) = p(1) - \int_0^1 p'(x)G(x)dx,$$

where the second equality uses the fact that $dp(x) = p'(x)dx$ by absolute continuity of p . Next, we have

$$\int_0^1 p'(x)G(x)dx = \int_0^1 p'(x)d \left(\int_0^x G(t)dt \right),$$

and we can use integration by parts for the Riemann-Stieltjes integral again to obtain

$$\begin{aligned} \int_0^1 p'(x)G(x)dx &= \left[p'(x) \left(\int_0^x G(t)dt \right) \right]_0^1 - \int_0^1 \left(\int_0^x G(t)dt \right) dp'(x) \\ &= p'(1) \left(\int_0^1 G(t)dt \right) - \int_0^1 \left(\int_0^x G(t)dt \right) dp'(x). \end{aligned}$$

Because G was arbitrary, the same transformations are true for $G = F$, and hence condition (3.2) is equivalent to

$$p'(1) \left(\int_0^1 G(t)dt \right) - \int_0^1 \left(\int_0^x G(t)dt \right) dp'(x) = p'(1) \left(\int_0^1 F(t)dt \right) - \int_0^1 \left(\int_0^x F(t)dt \right) dp'(x).$$

By condition (3.3), F and G have the same mean, and thus

$$p'(1) \left(\int_0^1 G(t)dt \right) = p'(1) \left(\int_0^1 F(t)dt \right)$$

which ends the proof. $\square$

By condition (3.3), F is a mean-preserving spread of G which implies that G second-order stochastically dominates F . Thus,

$$\int_0^x F(t)dt \geq \int_0^x G(t)dt, \quad \forall x \in [0, 1]. \quad (\text{A.14})$$

Because p is convex, p' is non-decreasing, and thus p' induces a positive measure. Therefore, condition (A.13) is satisfied if and only if $\int_0^x F(t)dt = \int_0^x G(t)dt$ for p' -almost all $x \in [0, 1]$. That is, the equality has to hold on every set that has positive measure under p' , in particular for each x at which there is a jump in p' , and for every interval on which p is strictly convex. We conclude that:

1. $\int_0^x F(t)dt = \int_0^x G(t)dt$ for each $x \in [a, b]$, for any interval $[a, b] \subset [0, 1]$ in which p is strictly convex;
2. p is affine in every interval $[a, b] \subset [0, 1]$ such that $\int_0^t F(x)dx > \int_0^t G(x)dx$ for all $t \in [a, b]$;
3. $\int_0^x F(t)dt = \int_0^x G(t)dt$ for each x at which p has a jump in the first derivative.

To strengthen the conclusion of points 1 and 3 above, we prove the following lemma.

Lemma 3. *If $x \in (0, 1)$ is such that $\int_0^x F(t)dt = \int_0^x G(t)dt$, then $F(x) = G(x)$.*

Proof. We will prove the contrapositive: if $F(x) \neq G(x)$ for some $x \in (0, 1)$, then $\int_0^x F(t)dt \neq \int_0^x G(t)dt$.

Fix $x \in (0, 1)$ and first suppose that $F(x) < G(x)$. Since F and G are right-continuous, there exists a $z > x$ such that $F(t) < G(t)$ for every $t \in (x, z)$. Then, since $\int_0^z F(t)dt \geq \int_0^z G(t)dt$ holds by equation (A.14), we have

$$\int_0^x F(t)dt = \int_0^z F(t)dt - \int_x^z F(t)dt > \int_0^z G(t)dt - \int_x^z G(t)dt = \int_0^x G(t)dt.$$

If instead $F(x) > G(x)$, since G is nondecreasing and F has full support and no atoms, there exists a $w < x$ such that $F(t) > G(t)$ for all $t \in (w, x)$. Then

$$\int_0^x F(t)dt = \int_0^w F(t)dt + \int_w^x F(t)dt > \int_0^w G(t)dt + \int_w^x G(t)dt = \int_0^x G(t)dt. \quad \square$$

By Lemma 3, $F(x) = G(x)$ in every interval in which p is strictly convex, and for every x at which p has a jump in the first derivative.

Take any maximal interval $[a, b]$ in which p is affine (that is, p is not affine on any other $[c, d]$ which contains $[a, b]$). By maximality, we must have $F(a) = G(a)$, $F(b) = G(b)$, and $\int_a^b F(t)dt = \int_a^b G(t)dt$ as we would otherwise violate the observation in the previous paragraph. Moreover, there exists $c \in [a, b]$ such that $u(c) = p(c)$ because otherwise, by condition (3.1), the function G would be constant on $[a, b]$ (while F is strictly increasing because the distribution has full support).

Now, take any interval $[a, b]$ where p is strictly convex. Then, $F(x) = G(x)$ for all $x \in [a, b]$, and because F has full support, $[a, b] \subseteq \text{supp}(G)$. Because G and p satisfy condition (3.1), we must have $u(x) = p(x)$ for all $x \in [a, b]$.

This ends the proof of claims (i) and (ii). We now prove two lemmas that will be used in the proof of the remaining part of the proposition.

Lemma 4. *Suppose that $p(x) > u(x)$ in some interval $[a, b]$. Then, $p(x)$ is piece-wise affine with at most one kink on $[a, b]$.*

Proof. Because F has full support, the function $\int_0^x F(t)dt$ is strictly convex. Because G satisfies condition (3.1), under the assumption of the lemma, it is not supported in $[a, b]$, and thus $\int_0^x G(t)dt$ is affine on $[a, b]$. Using inequality (A.14), we conclude that $\int_0^x F(t)dt$ and $\int_0^x G(t)dt$ coincide at at most one point in (a, b) , call it z^* . From condition 2 above, we obtain that p is affine in (a, z^*) and (z^*, b) . $\square$

Lemma 5. *Suppose that u is regular and let Π be the partition from condition (ii) of regularity. Then, for every interval $I \in \Pi$, the set $\{x \in \text{int}(I) : u(x) = p(x)\}$ is also an interval (potentially degenerate).*

Proof. The claim is obvious in the case when u is either affine or strictly concave on I . Consider the case when u is strictly convex. Let $\text{int}(I) = (a, b)$ and suppose that $\{x \in (a, b) : u(x) = p(x)\}$ is not an interval. Then, because u is upper semi-continuous, there exist $y_1, y_2 \in (a, b)$, $y_1 < y_2$, such that $u(y_1) = p(y_1)$, $u(y_2) = p(y_2)$ but $p(y) > u(y)$ for all $y \in (y_1, y_2)$. By Proposition 1, p must solve (3.7). However, this is a contradiction because the price function

$$p^*(x) = \begin{cases} p(x) & \text{if } x \notin (y_1, y_2) \\ u(x) & \text{if } x \in (y_1, y_2) \end{cases}$$

is feasible²¹ and achieves a strictly lower value of the objective in (3.7) (because F has full support). $\square$

We are ready to prove the last part of Proposition 2. By Lemmas 4 and 5, for every interval $I \in \Pi$, if $\text{cl}(I) = [a, b]$, there exists a finite partition of $[a, b]$ into subintervals (in fact, with not more than five elements) such that p is either affine or strictly convex on each subinterval. Because Π is finite, there exists a finite partition of $[0, 1]$ into intervals such that p is either affine or strictly convex on each interval in the partition. The cutoff points x_i can then be defined recursively.

A.5 Proof of Theorem 3

Let u be regular (Definition 1) and affine-closed (Definition 2), and let F be continuous. By Theorem 2, we can find G and p that satisfy conditions (3.1)–(3.3). From G we will build an optimal distribution of posterior means H which is induced by a monotone partitional signal. By Proposition 2, there are finitely many intervals $[x_i, x_{i+1}]$ in which p is either strictly convex or

²¹In particular, p^* is convex: It is enough to show that p^* has a supporting hyperplane at every x : For $x \notin (y_1, y_2)$, every supporting hyperplane of p at x must also support p^* ; and for $x \in (y_1, y_2)$, every (locally) supporting hyperplane of u at x must also support p^* . The above claim follows from two facts: (1) The subdifferential of a convex function is a non-decreasing correspondence; (2) If two convex functions u and p coincide at an interior point y and $p \geq u$ everywhere, then the subdifferential of u at y is a subset of the subdifferential of p at y .

affine. On each of these intervals $[a, b] = [x_i, x_{i+1}]$ we will verify that

$$\text{supp}(H) \cap [a, b] \subseteq \{x \in [a, b] : u(x) = p(x)\}, \quad (\text{A.15})$$

$$\int_a^b p(x) dH(x) = \int_a^b p(x) dF(x), \text{ and} \quad (\text{A.16})$$

$$F|_{[a,b]} \text{ is a mean-preserving spread of } H|_{[a,b]}. \quad (\text{A.17})$$

If (A.15), (A.16), and (A.17) hold on each interval, then for (H, p) , (3.1), (3.2) and (3.3) hold on $[0, 1]$. By Theorem 1, this is sufficient to verify optimality of H .

Consider each interval $[a, b] = [x_i, x_{i+1}]$ in turn, beginning with $[0, x_1]$. If $[a, b]$ is an interval where p is strictly convex, set $H(x) = G(x)$ for all $x \in [a, b]$. By Proposition 2, $u = p$ on $[a, b]$, hence (A.15) is satisfied. Since $F = G = H$ on $[a, b]$, (A.16) and (A.17) are automatically satisfied. The signal which induces H is full revelation of X on $[a, b]$, which is part of a monotone partitional signal (Definition 5).

If instead p is affine on $[a, b]$, let $y = \mathbb{E}[X|a \leq X \leq b]$. By Proposition 2, the mean of G conditional on $[a, b]$ is equal to y . If $u(y) = p(y)$, modify G by specifying that H puts all mass in the interval $[a, b]$ on y (pooling in the interval $[a, b]$). Formally, $H(x) = G(a)$ for $x \in [a, y]$ and $H(x) = G(b)$ for $x \in [y, b]$. Condition (A.15) holds because $u(y) = p(y)$. Condition (A.16) holds because p is affine on $[a, b]$ and H and F have the same conditional mean. Finally, condition (A.17) holds because $F|_{[a,b]}$ is a mean-preserving spread of $G|_{[a,b]}$, $G|_{[a,b]}$ is a mean-preserving spread of $H|_{[a,b]}$, and the mean-preserving spread relationship is transitive.

The remaining case is when p is affine on $[a, b]$ and $u(y) < p(y)$. Let $A := \{x \in [a, b] : u(x) = p(x)\}$. The support of G restricted to $[a, b]$ is a subset of A , by (3.1). Since u is upper semi-continuous and $u \leq p$, A is a closed subset of $[a, b]$. Since $y \notin A$ by assumption and y is the conditional mean of G on $[a, b]$, the support of G (hence its superset A) must contain points in $[a, y)$ and in $(y, b]$. Write A as the disjoint union $A = A_L \sqcup A_R$, where $A_L \subset [a, y)$ and $A_R \subset (y, b]$ are closed and nonempty.

Next we show that affine-closure implies that at least one of A_L and A_R is a closed interval that extends to a or b respectively; that is, either $A_L = [a, c]$, or $A_R = [d, b]$, or both, for some $c, d \in (0, 1)$. Suppose neither were true. Write the closed set A_L as a union of disjoint closed intervals, choose any one of these intervals that has its left endpoint not equal to a , and define α as its left endpoint. Similarly we can define $\beta < b$ as the right endpoint of an interval of A_R . By construction, the definition of affine-closure applies to u and p at α and β : they belong to A , so $u(\alpha) = p(\alpha)$ and $u(\beta) = p(\beta)$; $p \geq u$ at all points in $[a, b]$; and $p > u$ in a left-neighborhood and right-neighborhood of α and β respectively. Thus, by affine-closure, $u(x) = p(x)$ for all $\alpha \leq x \leq \beta$. In particular this holds for $x = y$, which contradicts our previous assumption that $u(y) < p(y)$.

From now on suppose that $A_L = [a, c]$ for some $c < y$; the symmetric case $A_R = [d, b]$ follows from the same argument. We now construct H on the interval $[a, b]$. Let $\delta := \min A_R$, and define ω as the smallest solution to $\mathbb{E}[X|\omega \leq X \leq b] = \delta$. The solution exists because $\mathbb{E}[X|\omega \leq X \leq b]$,

as a function of ω , is nondecreasing, continuous (F has no mass points), and ranges from $y < \delta$ at $\omega = a$ to $b > \delta$ at $\omega = b$.

Now consider the following monotone partition: $[a, \omega]$, $(\omega, b]$. The H that it induces has a mass point at $\gamma := \mathbb{E}[X|a \leq X \leq \omega]$ and one at δ . As it is a monotone partition of F , it satisfies (A.17).

Finally, we check that (3.1) holds. The required equality $u(\delta) = p(\delta)$ holds by construction, so the only thing left is to check that $u(\gamma) = p(\gamma)$. Since $A_L = [a, c]$, it is enough to check that $\gamma \leq c$. Suppose instead that $\gamma > c$. We will derive a contradiction by showing that this implies that F is not a mean-preserving spread of G in the interval $[a, b]$, contradicting Proposition 2.

When $\gamma > c$, all mass that G puts to the left of y must be to the left of γ . We show that this mass is smaller than what H puts: $G(\gamma) < H(\gamma)$. If instead $G(\gamma) \geq H(\gamma)$, then $G(z) \geq H(z)$ for all $z \in [a, \omega]$ with the inequality strict for at least some $z < \gamma$, since $H(a) = G(a)$ (because $H = G$ at all endpoints of the convex regions to the left of a , by construction) and both G and H are constant for $z \in (\gamma, \omega]$. Therefore $\int_a^\omega G(z)dz > \int_a^\omega H(z)dz$. The right-hand side evaluates to $\int_a^\omega H(z)dz = F(\omega)(\omega - \gamma) = \omega F(\omega) - \int_a^\omega z dF(z) = \int_a^\omega F(z)dz$, where the last equality is integration by parts. Putting things together, $\int_a^\omega G(z)dz > \int_a^\omega F(z)dz$, which contradicts the assumption that F is a mean-preserving spread of G .

Next, note that $\int_\omega^b H(z)dz = F(\omega)(\delta - \omega) + (b - \delta) = bF(b) - \omega F(\omega) - (1 - F(\omega))\delta = \int_\omega^b F(z)dz$, where the last equality is again by integration by parts. Furthermore, $1 = H(z) \geq G(z)$ for $z \in [\delta, b]$, and $H(z) > G(z)$ for $z \in [\gamma, \delta)$ (because both G and H are constant on $[\gamma, \delta)$ and $H(\gamma) > G(\gamma)$). Therefore $\int_\omega^b G(z)dz < \int_\omega^b H(z)dz = \int_\omega^b F(z)dz$. Since F is a mean-preserving spread of G , $\int_a^\omega G(z)dz \leq \int_0^\omega F(z)dz$. But then $\int_0^b G(z)dz = \int_a^\omega G(z)dz + \int_\omega^b G(z)dz < \int_a^\omega F(z)dz + \int_\omega^b F(z)dz = \int_a^b F(z)dz$, contradicting the assumption that F is a mean-preserving spread of G (which implies $\int_a^b G(z)dz = \int_a^b F(z)dz$).

This concludes the proof that (3.1) holds on $[a, b]$. Note that H is well-defined, as by construction $H = G$ on all endpoints of the intervals $[a, b]$. We conclude that H is optimal, and by construction it is induced by a monotone partitional signal.

We now prove the converse. If u is not affine-closed, then there exist $x, y \in (0, 1)$, $x < y$, and an affine function q such that: $u(x) = q(x)$, $u(y) = q(y)$; $q(z) \geq u(z)$ for all $z \in (x, y)$; there exists $w \in (x, y)$ such that $q(w) > u(w)$; and there exists $\varepsilon > 0$ such that $q(z) > u(z)$ for all $z \in (x - \varepsilon, x) \cup (y, y + \varepsilon)$, where ε is chosen so that $x - \varepsilon > 0$ and $y + \varepsilon < 1$.

Consider the distribution F that puts weight $\alpha > 0$ uniformly on $[x - \varepsilon, x - \varepsilon/2]$ and weight $1 - \alpha$ uniformly on $[y + \varepsilon/2, y + \varepsilon]$, where α is chosen so that $\mathbb{E}[F] = w$, where $q(w) > u(w)$ holds. That is, let F have density

$$f(z) = \begin{cases} 0 & \text{if } z < x - \varepsilon \\ \frac{2\alpha}{\varepsilon} & \text{if } x - \varepsilon \leq z \leq x - \frac{\varepsilon}{2} \\ 0 & \text{if } x - \frac{\varepsilon}{2} < z < y + \frac{\varepsilon}{2} \\ \frac{2(1-\alpha)}{\varepsilon} & \text{if } y + \frac{\varepsilon}{2} \leq z \leq y + \varepsilon \\ 0 & \text{if } z > y + \varepsilon \end{cases}.$$

Given such a prior F , the Sender cannot do strictly better than choosing a distribution of posterior means G that has support limited to x and y . To see this, first note that the mean-preserving spread condition implies that $\text{supp } G \subseteq [x - \varepsilon, y + \varepsilon]$. Suppose the Sender's utility were q . Since q is linear, all feasible distributions of posterior means are optimal; in particular, so is one with support equal to $\{x, y\}$. Since $q \geq u$ on $[x - \varepsilon, y + \varepsilon]$, this gives an upper bound to the utility attainable under u . The upper bound is attained if and only if all mass is concentrated at points where $q(z) = u(z)$. In particular, the distribution

$$G(z) = \begin{cases} 0 & \text{if } 0 \leq z < x \\ \alpha & \text{if } x \leq z < y \\ 1 & \text{if } y \leq z \leq 1 \end{cases}$$

is feasible and attains the upper bound.

To conclude, it is enough to show that no monotone partitional signal achieves the upper bound. First consider the trivial partition that pools all realizations at the prior mean $w = \mathbb{E}[F]$. By our choice of α , $u(w) < q(w)$, hence this is not optimal. Next consider monotone partitions with two or more signals. Since all prior mass is concentrated on $[x - \varepsilon, x - \varepsilon/2]$ and $[y + \varepsilon/2, y + \varepsilon]$, at least one of the intervals in the partition has a conditional posterior mean in $[x - \varepsilon, x - \varepsilon/2]$ or $[y + \varepsilon/2, y + \varepsilon]$. But by construction $u(z) < q(z)$ on such intervals. Hence the monotone partition is not optimal. We conclude that no monotone partition is optimal for such a prior F . $\square$

A.6 Material for Section 6.2 and proof of Proposition 4

We first present the parametrization of preferences that underlies Figures 8 and 9. Assume that the investor has CRRA utility v with Arrow-Pratt measure of relative risk aversion $\eta > 0$ over final wealth z ; that is, $v(z) = \frac{z^{1-\eta}-1}{1-\eta}$ if $\eta \neq 1$, and $v(z) = \ln z$ if $\eta = 1$. Given a posterior belief $x > \frac{1}{2}$ that the long position is profitable ($x < \frac{1}{2}$ is symmetric), the investor chooses the level of investment $y > 0$ in the risky asset that maximizes $xv(w + y - c) + (1 - x)v(w - y - c)$, or decides not to invest ($y = 0$) and receives the outside option $v(w)$. Let $y^*(x)$ be the optimal investment as a function of belief x .

Assume the analyst's payoff is proportional to the amount invested. Then (omitting the irrelevant proportionality constant) $u(x) \equiv y^*(x)$ can be shown to be

$$u(x) = \begin{cases} -(w - c) \frac{1 - \left(\frac{1-x}{x}\right)^{\frac{1}{\eta}}}{1 + \left(\frac{1-x}{x}\right)^{\frac{1}{\eta}}} & \text{if } x \leq x_0 \\ 0 & \text{if } x_0 < x < 1 - x_0 \\ (w - c) \frac{1 - \left(\frac{1-x}{x}\right)^{\frac{1}{\eta}}}{1 + \left(\frac{1-x}{x}\right)^{\frac{1}{\eta}}} & \text{if } x \geq 1 - x_0 \end{cases}$$

where x_0 is the threshold belief at which the investor is indifferent between investing or not: $v(w) = x_0 v(w + y^*(x_0) - c) + (1 - x_0) v(w - y^*(x_0) - c)$. If $\eta < 1$ (less risk averse than log utility), $u(x)$ is

strictly concave in $[0, x_0]$ and $[1 - x_0, 1]$; if $\eta > 1$ (more risk averse than log utility), then $u(x)$ is strictly convex in the same regions. For all $\eta > 0$, u is affine-closed. By Theorem 3, there exists a monotone partitional optimal signal. Since the multiplier $p(x)$ must be convex, hence continuous, $p(x)$ cannot coincide with $u(x)$ in a neighborhood of the discontinuities x_0 and $1 - x_0$. Thus, by Proposition 2, x_0 and $1 - x_0$ must be contained in pooling regions. These insights lead us to consider the multipliers depicted in Figure 8 and Figure 9. Define $w(x)$ as

$$w(x) = \left(-\mathbf{1}_{\{x \leq \frac{1}{2}\}} + \mathbf{1}_{\{x > \frac{1}{2}\}} \right) (w - c) \frac{1 - \left(\frac{1-x}{x}\right)^{\frac{1}{\eta}}}{1 + \left(\frac{1-x}{x}\right)^{\frac{1}{\eta}}}.$$

This coincides with u for $x \leq x_0$ and $x \geq 1 - x_0$. For $\eta < 1$, w is strictly concave on $[0, \frac{1}{2}]$ and $[\frac{1}{2}, 1]$. For $\eta > 1$, w is (globally) strictly convex.

To prove Proposition 4, we assume that

$$x_0 > \mathbb{E} \left[X \middle| X \leq \frac{1}{2} \right]. \quad (\text{A.18})$$

Proof of Proposition 4. In the concave case ($\eta < 1$), define $y = \mathbb{E}[X|X \leq \frac{1}{2}]$ and consider the piece-wise linear, convex function

$$p(x) = \begin{cases} u'(y)(x - y) + u(y) & \text{if } x \leq \frac{1}{2} \\ u'(1 - y)(x - (1 - y)) + u(1 - y) & \text{if } x > \frac{1}{2} \end{cases}.$$

By construction, p is tangent to u at y and $1 - y$, and has a kink at $\frac{1}{2}$. Furthermore, since p and u coincide at y and $1 - y$, and w is concave, $p(x) \geq w(x) \geq u(x)$ for all $x \in [0, 1]$ and $p(x) > u(x)$ for $x \notin \{y, 1 - y\}$. Consider the distribution of posterior means

$$G(x) = \begin{cases} 0 & \text{if } x < y \\ \frac{1}{2} & \text{if } y \leq x < 1 - y \\ 1 & \text{if } x \geq 1 - y \end{cases}$$

that puts atoms of size $F(\frac{1}{2}) = \frac{1}{2}$ on $y = \mathbb{E}[X|X \leq \frac{1}{2}]$ and on $1 - y = \mathbb{E}[X|X \geq \frac{1}{2}]$. Conditions (3.1)–(3.3) are satisfied, hence, by Theorem 1, G is optimal.

Now consider the convex case ($\eta > 1$). Consider the set of lines passing through $(x_0, u(x_0))$ with slopes m that satisfy $\frac{u(x_0) - u(0)}{x_0 - 0} < m < u'(x_0)$. Since $u(x) \leq w(x)$ and w is strictly convex, each such line intersects $u(x)$ at two points besides x_0 : $a(m) \in (0, x_0)$ and $b(m) \in (x_0, \frac{1}{2})$. Functions $a(\cdot)$ and $b(\cdot)$ are continuous and strictly increasing. Let $t(m) = \mathbb{E}[X|X \in [a(m), b(m)]]$. Since F is continuous and strictly increasing by assumption, $t(\cdot)$ is also continuous and strictly increasing. By assumption (A.18), $t\left(\frac{u(x_0) - u(0)}{x_0 - 0}\right) < \mathbb{E}[X|X \in [0, \frac{1}{2}]] < x_0$. By construction, $t(u'(x_0)) = \mathbb{E}[X|X \in [x_0, b(u'(x_0))]] > x_0$. By the intermediate value theorem, there exists a (unique) m^* such that

$t(m^*) = x_0$. Define $a^* := a(m^*)$, $b^* := b(m^*)$. Now consider the following p and G :

$$p(x) = \begin{cases} u(x) & \text{if } x < a^* \\ m^*(x - a^*) + u(a^*) & \text{if } a^* \leq x < b^* \\ u(x) & \text{if } b^* \leq x < 1 - b^* \\ -m^*(x - (1 - a^*)) + u(1 - a^*) & \text{if } 1 - b^* \leq x < 1 - a^* \\ u(x) & \text{if } x \geq 1 - a^* \end{cases},$$

$$G(x) = \begin{cases} F(x) & \text{if } x < a^* \\ F(a^*) + \mathbf{1}_{\{x \geq x_0\}}(F(b^*) - F(a^*)) & \text{if } a^* \leq x < b^* \\ F(b^*) + \mathbf{1}_{\{x \geq \frac{1}{2}\}}(F(1 - b^*) - F(b^*)) & \text{if } b^* \leq x < 1 - b^* \\ F(1 - b^*) + \mathbf{1}_{\{x \geq 1 - x_0\}}(F(1 - a^*) - F(1 - b^*)) & \text{if } 1 - b^* \leq x < 1 - a^* \\ F(x) & \text{if } x \geq 1 - a^* \end{cases}.$$

G reveals x when $x < a^*$ or $x \geq 1 - a^*$. The remaining intervals $[a^*, b^*]$, $(b^*, 1 - b^*)$, and $[1 - b^*, 1 - a^*]$ are pooled at x_0 , $\frac{1}{2}$ and $1 - x_0$, respectively. Conditions (3.1)–(3.3) are satisfied by construction, hence, by Theorem 1, G is optimal. $\square$

A.7 Proof of Theorem 4

Because the proof is similar to the proof of Theorem 1 and Theorem 2, we omit the details and only highlight the differences.

Verifying that H is unimprovable amounts to checking that H is a solution to the following optimization problem:

$$\max_G \int_0^1 u(x) dG(x) \tag{A.19}$$

subject to

$$F \text{ is a mean-preserving spread of } G; \tag{A.20}$$

$$G \text{ is a mean-preserving spread of } H, \tag{A.21}$$

where the second condition reflects the constraint that the Sender can only disclose additional information but cannot “hide” the information revealed according to distribution H . We can write down the analog of Theorem 1 for this problem: If there exist convex functions p and q , and a distribution G such that (A.20) – (A.21) hold, and

$$\text{supp}(G) \subseteq \text{argmax}_{x \in [0, 1]} \{u(x) - p(x) + q(x)\} \tag{A.22}$$

$$\int_0^1 p(x) dG(x) = \int_0^1 p(x) dF(x), \tag{A.23}$$

$$\int_0^1 q(x) dG(x) = \int_0^1 q(x) dH(x), \quad (\text{A.24})$$

then G is optimal for the problem (A.19) – (A.21). The additional price function q and the additional constraint (A.24) are a consequence of adding the constraint that G is a mean-preserving spread of H . This result is proven in the same way as Theorem 1. (Condition (A.22) is equivalent to assuming $p(x) - q(x) \geq u(x)$ and $\text{supp}(G) \subseteq \{x \in [0, 1] : u(x) = p(x) - q(x)\}$.)

With this verification tool, we can write down the conditions under which H is an optimal solution to the problem (A.19) – (A.21):

$$\text{supp}(H) \subseteq \text{argmax}_{x \in [0, 1]} \{u(x) - p(x) + q(x)\} \quad (\text{A.25})$$

$$\int_0^1 p(x) dH(x) = \int_0^1 p(x) dF(x), \quad (\text{A.26})$$

where condition (A.24) is omitted because it holds vacuously. Because q only appears in condition (A.25), we can define $\hat{u} = u + q$ which leads to a function that is a convex translation of u . This proves the “if” part of Theorem 4: If the conditions of Theorem 4 hold, we can define $q = \hat{u} - u$, and conditions (A.25) – (A.26) will hold. Condition (A.25) is equivalent to (3.1) with u replaced by $\hat{u}$ because prices can be normalized appropriately. Thus, H is unimprovable.

To prove the “only if” part, it is enough to prove existence of convex functions q and p such that conditions (A.22) – (A.24) hold. This can be done analogously as in the proof of Theorem 2, where the additional function q is derived in the same way as the function p from the additional constraint (A.21).

There are two differences in the proof. First, the generalized Slater condition (A.4) is not guaranteed to hold for the problem in which G is “sandwiched” between H and F in the mean-preserving spread order. This can be circumvented by studying a slightly different perturbation than in the proof of Theorem 2: The mean-preserving spread condition is only imposed on $[\epsilon, 1 - \epsilon]$ and additionally G is only required to be a mean-preserving spread of H_ϵ (instead of H) with the property that H_ϵ converges to H in the weak* topology as $\epsilon \rightarrow 0$, and F is a *strict* mean-preserving spread of H_ϵ on $[\epsilon, 1 - \epsilon]$ in the sense that

$$\begin{aligned} \int_0^x F(t) dt &> \int_0^x H_\epsilon(t) dt, \quad \forall x \in [\epsilon, 1 - \epsilon], \\ \int_x^1 H_\epsilon(t) dt &> \int_x^1 F(t) dt, \quad \forall x \in [\epsilon, 1 - \epsilon]. \end{aligned}$$

Such an approximation H_ϵ exists because the support of F contains 0 and 1 by assumption. The perturbed problem satisfies the Slater condition (A.4), and the rest of the proof is fully analogous.

Second, it is not possible to take the final step in the proof of Theorem 2, that is, extend the conclusion to discontinuous utility functions. This is because we cannot guarantee that the sequence of prices for the perturbed problem will have a converging subsequence. Constraint (A.24) might force the prices for the perturbed problem to have an arbitrarily high slope, and thus the sequence

might diverge.

This is not just a problem with the proof. The “only if” part of Theorem 4 is false in the case of discontinuous utility functions. To see this, take an example where the utility function u of the Sender has a discontinuity in the interior of $[0, 1]$, and consider $H = F$ (this would arise if the Sender competes against another Sender with a strictly convex utility function). Because full disclosure is unimprovable, if the result were true, there would exist a convex translation $\hat{u}$ of u such that $\hat{u}$ would coincide with some convex p on $[0, 1]$ (this is necessary to support a fully-revealing solution). But this is impossible because any $\hat{u}$ would still have a discontinuity in the interior of $[0, 1]$, and every convex function is continuous on the interior of its domain. Therefore, the “only if” part of Theorem 4 is only established for the case of regular, continuous utility functions.

A.8 Proof of Proposition 5

Lemma 6. *Suppose that $\hat{u}_i$ is a convex translation of u_i . If p_i is a convex function satisfying all conditions of Theorem 4 (with some distribution H), then*

$$p_i(x) = \begin{cases} \hat{u}_i(x) & x \leq a_i \\ \hat{u}_i(a_i) + (x - a_i)\hat{u}_i(b_i) & a_i \leq x \leq b_i, \\ \hat{u}_i(x) & b_i \leq x \end{cases}$$

for some $a_i \leq c_i \leq b_i$.

Proof. Because the prior mean is smaller than c_i , $\hat{u}_i$ has to coincide with p_i for at least some points in $[0, c_i)$, by condition (A.22). Because $\hat{u}_i$ is convex on $[0, c_i]$ (because u_i is convex), the only possibility is that $\hat{u}_i$ coincides with p_i on some subinterval $[\underline{d}, a_i]$ of $[0, c_i]$ (this follows from the properties proved in Proposition 2). Moreover, we must have $\underline{d} = 0$ as otherwise the mean-preserving spread condition would be violated on $[0, \underline{d}]$, again by Proposition 2. By the same reasoning, if $\hat{u}_i$ coincides with p_i anywhere in the interval $(c_i, 1]$, then it has to coincide on a subinterval of the form $[b_i, 1]$. Finally, by Proposition 2 and due to the shape of $\hat{u}_i$, the price function has to be affine in the remaining interval $[a_i, b_i]$ because it can touch $\hat{u}_i$ only at c_i , if at all. $\square$

Lemma 6 restricts the set of distributions that can emerge as optimal for any optimization problem of Sender i . Because $\hat{u}_i$ is strictly convex on $[0, a_i]$, by Proposition 2, the state is fully revealed in that interval. In the interval $[a_i, c_i]$, there can only be an atom in the posterior distribution at c_i (if p_i touches $\hat{u}_i$ at c_i). Finally, because $\hat{u}_i$ may be affine on $[c_i, 1]$, the posterior distribution can have mass anywhere in this interval, as long as the mean-preserving spread condition is satisfied. This leads us to consider the following class $\mathcal{H}_i$ of posterior distributions for Sender i : any $H \in \mathcal{H}_i$ reveals the state fully on $[0, a_i]$ for some $a_i \leq c_i$, has no mass on (a_i, c_i) , and satisfies $\mathbb{E}_H[X|X \geq a_i] = \mathbb{E}_F[X|X \geq a_i]$. By the above reasoning, the class $\mathcal{H}_i$ contains all

candidate distributions that may be unimprovable for Sender i . We will show that indeed each $H \in \mathcal{H}_i$ is unimprovable for i .

To this end, given any $H \in \mathcal{H}_i$, we will construct a $\hat{u}_i$ and p_i to satisfy conditions (A.22) – (A.24). The function $\hat{u}_i$ is constructed by adding a wedge function $w(x) = \alpha(x - c_i)^+$ to u_i . Consider a function p_i that coincides with u_i on $[0, a_i]$, and is affine on $[a_i, 1]$, tangent to u_i at c_i . Such a function exists because u_i has bounded slope and is not convex (there is a kink at c_i). Then, choose α in the wedge function w so that $\hat{u}_i = u_i + w$ coincides with p_i on $[c_i, 1]$. It is now a routine check that all conditions (A.22) – (A.24) hold.

In the final step, in order to find the set of equilibrium distributions, we only have to take the intersection of $\mathcal{H}_i$ over $i \in N$. This intersection is equal to $\mathcal{H}_1$ because these sets are nested (by assumption, c_i is highest for Sender 1).

Thus, the set of equilibrium distributions contains exactly H such that H is feasible, fully reveals the state below some $a \leq c_1$, and has no mass on (a, c_1) .

The least informative equilibrium $\underline{H}$ is thus as follows: a is chosen so that the conditional mean of F conditional on $X \geq a$ is equal to c_1 . Then, $\underline{H}$ reveals the state below a and puts a mass point at c_1 . Hence, this is just the individually-optimal persuasion scheme of Sender 1.

A.9 Proof of Proposition 6

Fix $\alpha > -1$. Consider the price function $p(x, y) = \max\{p_{12}(x, y), p_1(x, y), p_2(x, y), p_\emptyset(x, y)\}$, where $p_{12}(x, y), p_1(x, y), p_2(x, y), p_\emptyset(x, y)$ are affine functions that are defined next. Let $A_{12} \subset [0, 1]^2$ be the region where $p(x, y) = p_{12}(x, y)$, and define A_1, A_2 , and $A_\emptyset$ similarly.

Let $p_{12}(x, y) = ax + a'y + a''$. The region A_{12} will be the set of realizations of X that (pooled together) induce the agent to take both actions. Thus, it is natural to conjecture that this region will have its center of mass at (β, β) , and that, by symmetry, $a = a'$. Thus, $p_{12}(\beta, \beta) = 2a\beta + a'' = 2 + \alpha = u(\beta, \beta)$. Therefore $p_{12}(x, y) = a(x + y) + 2 + \alpha - 2a\beta$.

Similarly, $p_1(x, y)$ is chosen to determine the region A_1 , the set of realizations of X that induce the agent to take action 1 only. Since we need $p \geq u$ and $p = u$ at some point in the region, $p_1(x, y)$ must have the form $p_1(x, y) = bx + b'$. For $p_1 = u$ to hold at the threshold β , $p_1(\beta, y) = b\beta + b' = 1$. Therefore $p_1(x, y) = bx + 1 - b\beta$. By symmetry, $p_2(x, y) = by + 1 - b\beta$. Finally, $p_\emptyset(x, y) \equiv 0$.

There are two free parameters: a and b . We now impose the condition that the affine regions of $p(x, y)$ are such that $p = u$ at their centers of mass.

Let $s(a, b) := \frac{\int_{A_{12}} x dF(x, y)}{\int_{A_{12}} dF(x, y)}$ be the x component of the center of mass of A_{12} . (By symmetry of F and A_{12} , $s(a, b)$ is also the y component.) Similarly, define $t(a, b) := \frac{\int_{A_1} x dF(x, y)}{\int_{A_1} dF(x, y)}$ to be the x component of the center of mass of region A_1 . By symmetry, it is also the y component of the center of mass of A_2 .

To apply Theorem 1b, we need to find parameters a and b such that $s(a, b) = \beta$ and $t(a, b) = \beta$. It is in general difficult to write $s(a, b)$ and $t(a, b)$ explicitly, even for simple choices of the joint distribution F . However, it is sufficient for our purposes to prove that the system of equations

admits a solution.

Lemma 7. *There exist a^* and b^* such that $s(a^*, b^*) = \beta$ and $t(a^*, b^*) = \beta$.*

The proof is in Appendix A.12.3; a sketch follows. When the slope b of p_1 is very small, the center of mass of A_1 falls below $x = \beta$; conversely, when the slope is very large, the center of mass falls above $x = \beta$. A similar pattern holds for slope a and the center of mass of A_{12} . The centers of mass are continuous in the slopes. By the Poincaré-Miranda theorem (a multidimensional version of the Intermediate Value Theorem which is equivalent to Brouwer's Fixed Point Theorem), we show that there exist slopes a^* and b^* such that $s(a^*, b^*) = \beta$ and $t(a^*, b^*) = \beta$.

The assumptions on β and F (beyond symmetry) are only used to prove Lemma 7, and are thus not necessary assumptions for this approach.

We can now apply Theorem 1b. Let Y be the random variable on $[0, 1]^2$ that is built from X as follows: for every realization of X , if $X \in A \in \{A_{12}, A_1, A_2, A_\emptyset\}$, then Y is equal to the center of mass of A . (With probability 1, X belongs to exactly one of the four sets.) Then, by construction, $Y = E[X|Y]$ holds. Let G be the distribution of Y . By Theorem 3.4.2(a) in Müller and Stoyan (2002), $Y = E[X|Y]$ implies $G \leq_{cx} F$. Thus, condition (7.3) is satisfied.

By the choice of a^* and b^* , the centers of mass of A_{12} , A_1 , A_2 are located respectively at (β, β) , (β, y) for some $y \in (0, \beta)$, and (x, β) for some $x \in (0, \beta)$, points at which $p = u$ by construction of p . Lastly, $p \equiv u$ on $A_\emptyset$, if present. Thus, condition (7.1) is satisfied.

Next, note that

$$\int_{A_{12}} p(x, y) dF(x, y) = \int_{A_{12}} p_{12}(x, y) dF(x, y) = \int_{A_{12}} p_{12}(x, y) dG(x, y)$$

because p_{12} is affine on A_{12} . The argument for the other regions is identical. Putting them together verifies condition (7.2).

Finally, as the envelope of hyperplanes, p is convex, and $p \geq u$ by construction. Therefore Theorem 1b applies, and G is optimal.

A.10 Proof of Claim 1

If τ^* is the optimal solution for some prior μ_0 , and $P : \Delta(\Theta) \rightarrow \mathbb{R}$ is outer-convex and $P \geq V$, then

$$\begin{aligned} \int_{\Delta(\Theta)} V(\mu) d\tau^*(\mu) &\stackrel{(1)}{\leq} \int_{\Delta(\Theta)} P(\mu) d\tau^*(\mu) \stackrel{(2)}{\leq} \int_{\Delta(\Theta)} \int_{\Theta} P(\delta_\theta) d\mu(\theta) d\tau^*(\mu) \\ &\stackrel{(3)}{=} \int_{\Theta} P(\delta_\theta) \int_{\Delta(\Theta)} \mu(\theta) d\tau^*(\mu) d\theta \stackrel{(4)}{=} \int_{\Theta} P(\delta_\theta) d\mu_0(\theta), \end{aligned} \quad (\text{A.27})$$

where (1) follows from $V \leq P$, (2) from the fact that P is outer-convex, (3) is by change of the order of integration, and (4) follows from condition (8.5). On the other hand,

$$\int_{\Delta(\Theta)} V(\mu) d\tau^*(\mu) = \int_{\Delta(\Theta)} P^*(\mu) d\tau^*(\mu) = \int_{\Theta} P^*(\delta_\theta) d\mu_0(\theta),$$

where the first equality follows from condition (8.3) and the second from condition (8.4).

Moreover, $\text{co}(V)(\mu_0) = \int_{\Delta(\Theta)} V(\mu) d\tau^*(\mu)$ follows from the results of [Kamenica and Gentzkow \(2011\)](#).

A.11 Proof of Proposition 7

In the proof, we normalize $\bar{e} = 1$ to simplify exposition (the proof is identical for a general $\bar{e}$).

First, note that the function $v(x, y)$ has an “edge” of non-differentiability described by the equation $y = \bar{y} \equiv 2$ (using the normalization $\bar{e} = 1$). Let $h_\gamma(x, y)$ denote a family of hyperplanes, parametrized by $\gamma > 0$, that coincide with $v(x, y)$ along $y = \bar{y}$:

$$h_\gamma(x, y) = (1 - \beta) [x + \gamma(y - \bar{y})].$$

When $\gamma > 0$, $h_\gamma(x, y)$ lies strictly above $v(x, y)$ for $y > \bar{y}$.

Second, we find all points (x, y) with $y < \bar{y}$ at which $v(x, y)$ and $h_\gamma(x, y)$ coincide. By a simple calculation, these points are described by the equation $x = 2\gamma$. Moreover, $v(x, y) \geq h_\gamma(x, y)$ when $y < \bar{y}$ and $x \leq 2\gamma$, and $v(x, y) \leq h_\gamma(x, y)$ otherwise.

Third, we construct the price function. Define

$$p_\gamma(x, y) = \max \{v(x, y), h_\gamma(x, y)\} = \begin{cases} v(x, y) & x \leq 2\gamma, y \leq \bar{y} \\ h_\gamma(x, y) & \text{otherwise} \end{cases}.$$

Let x^* be defined as in the statement of Proposition 7. We choose γ^* so that v and h_γ intersect exactly at $x = x^*$, that is, we set $\gamma^* = x^*/2$. Finally, define $P(\mu) \equiv p_{\gamma^*}(\mathbb{E}_\mu X, \mathbb{E}_\mu u_A(\beta X))$. From now on, we fix $\gamma = \gamma^*$ and drop it from all the subscripts.

Fourth, it is immediate that $P(\mu) \geq V(\mu)$ for all μ because $p(x, y) \geq v(x, y)$ for all (x, y) . The following result is the most technical step in the proof, and hence we relegate its proof to the Technical Appendix [A.12](#).

Lemma 8. *The function P is outer-convex.*

Proof. See Appendix [A.12.4](#). □

Fifth, we verify conditions (8.3) – (8.5) of Theorem 5 with τ^* defined as in Proposition 7. Formally,

$$\tau^*(F|_{[x^*, \bar{x}]}) = 1 - F(x^*),$$

and

$$\tau^*(\{\delta_x\}_{x \in A}) = \int_A dF(x),$$

for any measurable $A \subset [\underline{x}, x^*]$.

Condition (8.3) follows because along the curve $(x, u_A(\beta x))$, for $x \in [\underline{x}, x^*]$, by definition, p and v coincide. Moreover, at $\mu = F|_{[x^*, \bar{x}]}$, we have $y = \mathbb{E}_{F|_{[x^*, \bar{x}]}}[u_A(\beta X)] = \bar{y}$, by the choice of x^* ,

so $P(\mu) = V(\mu)$.

Condition (8.4) follows because

$$\int_{\Delta([\underline{x}, \bar{x}])} P(\mu) d\tau^*(\mu) = \int_{\underline{x}}^{x^*} P(\delta_x) dF(x) + P(F|_{[x^*, \bar{x}]}) (1 - F(x^*)) = \int_{\underline{x}}^{\bar{x}} P(\delta_x) dF(x),$$

where the last equality is a consequence of the fact that P is affine in the relevant region.

Finally, condition (8.5) holds trivially by the choice of τ^* .

A.12 Technical Appendix

In this Appendix, we fill the gaps in the proof of Theorem 2, prove Lemma 7 used in the proof of Proposition 6, and prove Lemma 8 used in the proof of Proposition 7.

A.12.1 Step 2 without Assumption 1

For a fixed ϵ , consider a pair (G, p) from Step 1 of the proof of Theorem 2 from Appendix A.2. We can assume without loss of generality that p is non-negative.²² We first show how to modify the function p to ensure that its slope is uniformly bounded (this will be important in the next step of the proof). When we consider the sequence p_n , the slope of p_n could diverge to infinity close to the endpoints, upsetting uniform convergence. We show in the lemma below that we can control the slope by using the properties of the pair (G, p) established in Step 1.

Lemma 9. *Consider problem (A.1) - (A.3) for a fixed $\epsilon > 0$. There exists a convex function q which satisfies (A.7) and (A.9) and has a slope uniformly bounded by*

$$\max \left\{ \frac{c}{\int_0^{\underline{x}} (\underline{x} - x) dF(x)}, \frac{c}{\int_{\bar{x}}^1 (x - \bar{x}) dF(x)} \right\},$$

where c is a constant that does not depend on ϵ , and

$$\underline{x} := \inf \{x \in [0, 1] : u(x) = p(x)\},$$

$$\bar{x} := \sup \{x \in [0, 1] : u(x) = p(x)\}.$$

Proof. In the case when either $\underline{x} = 0$ or $\bar{x} = 1$, there is nothing to prove because the bound is equal to ∞ . We assume otherwise, and show that the slope of q can be uniformly bounded by $c / \int_0^{\underline{x}} (\underline{x} - x) dF(x)$ for $x \in [0, 1/2]$ (an analogous argument establishes the other bound for $x \in [1/2, 1]$).

Recall that u is assumed continuous, hence bounded, and that by the regularity assumption u has a slope uniformly bounded by some $M < \infty$. Because p is convex, $p \geq u$, and p coincides with u at $\underline{x}$ and $\bar{x}$, p inherits the bound M on the slope in the interval $[\underline{x}, \bar{x}]$.

²²The optimization problem is unaffected by adding a constant to the utility function u , so we can assume without loss of generality that $u(x) \geq 0$ for all $x \in [0, 1]$, and then non-negativity of p follows from $p \geq u$.

Next, note that condition (A.9) is not affected by modifying $p(x)$ on $[0, \underline{x}]$ or $(\bar{x}, 1]$. Because G puts no mass on $[0, \underline{x}] \cup (\bar{x}, 1]$ by property (A.9), the value of the integral $\int_0^1 p(x)dG(x)$ is also unaffected. Therefore, we can replace p on $[0, \underline{x}]$ by some other function q and preserve conditions (A.7) and (A.9) as long as

$$\int_0^{\underline{x}} p(x)dF(x) = \int_0^{\underline{x}} q(x)dF(x). \quad (\text{A.28})$$

Consider a function q that coincides with p on $[\underline{x}, 1]$, and is affine otherwise: $q(x) = p(\underline{x}) + \Delta(\underline{x} - x)$, for some $\Delta > 0$, for $x \in [0, \underline{x}]$. Choose Δ to satisfy equation (A.28). Because p is convex, it is clear that such Δ has to be larger than the slope of p at $\underline{x}$, so q remains convex. Moreover, we have

$$\begin{aligned} \|u\|_\infty &\geq \int_0^1 u(x)dG(x) \stackrel{(1)}{=} \int_0^1 p(x)dG(x) \stackrel{(2)}{=} \int_0^1 p(x)dF(x) \stackrel{(3)}{\geq} \int_0^{\underline{x}} p(x)dF(x) \\ &\stackrel{(4)}{=} \int_0^{\underline{x}} q(x)dF(x) \stackrel{(5)}{\geq} \Delta \int_0^{\underline{x}} (\underline{x} - x)dF(x), \end{aligned} \quad (\text{A.29})$$

where (1) follows from (A.9), (2) follows from (A.7), (3) follows from the fact that p is non-negative, and (4) and (5) from the definition of q . We conclude that

$$\Delta \leq \frac{\|u\|_\infty}{\int_0^{\underline{x}} (\underline{x} - x)dF(x)}.$$

Therefore, the slope of q is bounded by

$$\frac{\max\{\|u\|_\infty, M\}}{\int_0^{\underline{x}} (\underline{x} - x)dF(x)}.$$

This finishes the proof.²³ □

We come back to the proof of Step 2 without Assumption 1. Recall that we have a sequence (G_n, p_n) satisfying (A.2), (A.3), (A.7) and (A.9), with each p_n convex. Moreover, by Lemma 9, we can modify the functions p_n so that p_n has a slope bounded uniformly by

$$\max \left\{ \frac{c}{\int_0^{\underline{x}_n} (\underline{x}_n - x)dF(x)}, \frac{c}{\int_{\bar{x}_n}^1 (x - \bar{x}_n)dF(x)} \right\},$$

where c does not depend on n , and $\underline{x}_n$ and $\bar{x}_n$ are defined by Lemma 9:

$$\underline{x}_n := \inf\{x \in [0, 1] : u(x) = p_n(x)\},$$

$$\bar{x}_n := \sup\{x \in [0, 1] : u(x) = p_n(x)\}.$$

We can assume without loss (by passing to a subsequence if necessary) that both $\underline{x}_n$ and $\bar{x}_n$ converge

²³ We have not verified that such q satisfies $q \geq u$ on $[0, \underline{x}]$. However, this does not pose a problem for the remainder of the proof because we will only need the bound derived in the lemma to hold for $\underline{x}$ sufficiently close to 0. If q does not satisfy $q \geq u$, we can always replace $\underline{x}$ with some smaller $\underline{x}' > 0$, and the rest of the proof remains the same.

to some $\underline{x}$ and $\bar{x}$, respectively. If $\underline{x} > 0$ and $\bar{x} < 1$, then, for sufficiently high n , all p_n have a slope uniformly bounded by

$$\max \left\{ \frac{c}{\int_0^{\underline{x}/2} (\underline{x}/2 - x) dF(x)}, \frac{c}{\int_{(\bar{x}+1)/2}^1 (x - (\bar{x}+1)/2) dF(x)} \right\},$$

using the assumption that 0 and 1 are in the support of F . Consider the opposite case when either (i) $\underline{x} = 0$ or (ii) $\bar{x} = 1$. Then, for a sufficiently small $\delta > 0$, all p_n have a uniformly bounded slope on $[\delta, 1 - \delta]$, for sufficiently high n . This is because each $p_n \geq u$, p_n is convex, and thus p_n has a slope bounded by the slope of u (which is bounded by M by the regularity assumption).

We can thus conclude that for every (small) $\delta > 0$, p_n have a uniformly bounded slope on $[\delta, 1 - \delta]$. Thus, the sequence of functions p_n is uniformly bounded on $[\delta, 1 - \delta]$. This follows from the fact that each p_n is convex, has a uniformly bounded slope, and the domain $[\delta, 1 - \delta]$ is compact. A uniformly bounded sequence of convex functions is Lipschitz continuous with a common Lipschitz constant L . In particular, the sequence $(p_n)_n$ is equi-continuous on $[\delta, 1 - \delta]$. By the Arzelà-Ascoli Theorem, p_n has a uniformly converging subsequence on every interval $[\delta, 1 - \delta]$. Therefore, a subsequence of p_n converges to some continuous convex p on $(0, 1)$, uniformly on each compact subset of $(0, 1)$. We can complete the definition of p by specifying that p is continuous at 0 and at 1 (the properties of p at any single point do not play a role).

Just as in Step 2 of the proof from Appendix A.2, we prove that G_n converges in the weak* topology to some $G \in \Delta([0, 1])$, and that the limiting pair (G, p) satisfies conditions (3.1) and (3.3). A separate argument is needed to show condition (3.2) because now we only have convergence of p_n to p uniformly on every compact subset of $(0, 1)$ but not necessarily on $[0, 1]$.

Define, for each n , the smallest convex function q_n that coincides with p_n on $[\underline{x}_n, \bar{x}_n]$. Note that on $[\underline{x}_n, \bar{x}_n]$ the slope of p_n is bounded by the slope of u (which is bounded by M by assumption), so q_n can be constructed by linearly extending p_n beyond $[\underline{x}_n, \bar{x}_n]$ with the slope equal to the relevant derivative of p_n at the endpoints $\underline{x}_n$ and $\bar{x}_n$. Obviously, we have $p_n \geq q_n$. By construction, q_n has a uniformly bounded slope on $[0, 1]$ (bounded by M), so by the same argument as above, it has a uniformly convergent subsequence to some function q . Therefore (indexing the subsequence by n again), $\int_0^1 q_n(x) dG_n(x) \rightarrow \int_0^1 q(x) dG(x)$. To see this, note that

$$\int_0^1 q_n dG_n - \int_0^1 q dG = \int_0^1 (q_n - q) dG_n - \int_0^1 q (dG - dG_n);$$

The second integral converges to zero by the definition of convergence of G_n to G in the weak* topology. For the first integral, we have

$$\int_0^1 (q_n - q) dG_n \leq \sup_{x \in [0, 1]} \{|q_n(x) - q(x)|\}$$

which converges to zero because q_n converges to q uniformly on $[0, 1]$.

We have

$$\begin{aligned}
\int_0^1 p(x) dF(x) &\stackrel{(1)}{\geq} \int_0^1 p(x) dG(x) \stackrel{(2)}{\geq} \int_0^1 q(x) dG(x) = \lim_n \int_0^1 q_n(x) dG_n(x) \\
&\stackrel{(3)}{=} \lim_n \int_0^1 p_n(x) dG_n(x) \stackrel{(4)}{=} \lim_n \int_0^1 p_n(x) dF(x) \stackrel{(5)}{\geq} \lim_n \int_\delta^{1-\delta} p_n(x) dF(x) \\
&\stackrel{(6)}{=} \int_\delta^{1-\delta} p(x) dF(x) \stackrel{(7)}{\geq} \int_0^1 p(x) dF(x) - \varepsilon(\delta), \quad (\text{A.30})
\end{aligned}$$

where (1) follows because p is convex and F is a mean-preserving spread of G , (2) follows because the inequality $p_n(x) \geq q_n(x)$ is preserved in the limit, (3) follows because, by definition, p_n and q_n coincide on the support of G_n , (4) follows from (A.7), (5) follows for any $\delta > 0$ from non-negativity of p_n , (6) follows because p_n converges to p uniformly on every compact subset of $(0, 1)$, and (7) is true for some $\varepsilon(\delta)$ which goes to zero as $\delta \rightarrow 0$. Because δ (and hence $\varepsilon(\delta)$) can be arbitrarily small, we must have:

$$\int_0^1 p(x) dF(x) = \int_0^1 p(x) dG(x),$$

which is what we wanted to prove. This finishes the proof of Step 2, i.e. we have shown conditions (3.1)–(3.3) for the case of a continuous u (and hence the optimality of G , by Theorem 1).

A.12.2 Proof of Lemma 1

First, we note that Lemma 2 stated in the proof of Proposition 2 holds even without the additional assumptions on F made in Proposition 2 (the proof of Lemma 2 does not use these assumptions). Thus, we can use properties 1 – 3 of (G, p) stated in Appendix A.4 after Lemma 2 even though Proposition 2 itself may not hold without the additional assumptions on F .

We prove that the functions in the sequence p_ϵ are uniformly bounded. Suppose not. Then there exists a subsequence of p_ϵ (which we take to be the original sequence to simplify notation) such that $\lim_{\epsilon \rightarrow 0} \|p_\epsilon\|_\infty = \infty$. By the assumption that u has a uniformly bounded slope in the intervals where it is continuous, the only possibility is that p_ϵ has an affine piece whose slope diverges to infinity and which touches u at one of the points of discontinuity y_i . Because there are finitely many points of discontinuity of u , we can choose a divergent subsequence (which we take to be the sequence itself) in which each p_ϵ touches u with an affine piece at the same discontinuity point y_i .

Using the properties of u and $\bar{u}_\epsilon$, we claim that for small enough ϵ the affine piece of p_ϵ that touches u at y_i must have the following properties: (i) p_ϵ is affine on $[\delta_\epsilon, 1]$ for some $\delta_\epsilon \leq y_i$, and is not affine on any interval that strictly contains $[\delta_\epsilon, 1]$, and (ii) p_ϵ touches $\bar{u}_\epsilon$ only at y_i in the interval $[y_i, 1]$, and (iii) $\delta_\epsilon \rightarrow y_i$ as $\epsilon \rightarrow 0$. Note that condition (i) takes the above form because we have assumed that the function u jumps up at y_i (if it jumped down, we would work with the interval $[0, \delta_\epsilon]$ instead of $[\delta_\epsilon, 1]$).

We argue why properties (i) – (iii) hold. Properties (i) and (ii) hold because, by the choice of the sequence p_ϵ , the affine piece of p_ϵ that touches u at y_i has a divergent slope. Because u has a

uniformly bounded slope whenever it is continuous, and only finitely many (interior) discontinuities, for small enough ϵ , p_ϵ cannot touch $\bar{u}_\epsilon$ to the right of y_i . Because 1 belongs to the support of F , $\int_0^1 F(x)dx = \int_0^1 G(x)dx$, and condition (3.1) implies that G has no mass to the left of y_i , it follows that $\int_0^t F(x)dx > \int_0^t G(x)dx$ for all $t > y_i$. Property 2 of (G, p) stated in Appendix A.4 then implies that p_ϵ is affine to the right of y_i . We can define $\delta_\epsilon \leq y_i$ by requiring that $[\delta_\epsilon, 1]$ is the maximal interval in which p_ϵ is affine (no interval on which p_ϵ is affine strictly contains it). Property (iii) then follows from the fact that the slope of the affine piece of p_ϵ on $[\delta_\epsilon, 1]$ goes to plus infinity but p_ϵ touches $\bar{u}_\epsilon$ at y_i .

We are ready to obtain a contradiction. By the properties 1 – 3 of (G, p) stated in Appendix A.4, and the definition of δ_ϵ , F is a mean-preserving spread of G_ϵ in the interval $[\delta_\epsilon, 1]$. Because p_ϵ does not touch u to the right of y_i , G_ϵ must put all mass on $[\delta_\epsilon, y_i]$, and hence it has to be that $\mathbb{E}[X|X \geq \delta_\epsilon] \in [\delta_\epsilon, y_i]$. In the limit as $\epsilon \rightarrow 0$, we obtain $\mathbb{E}[X|X \geq y_i] = y_i$ which is a contradiction because 1 belongs to the support of F and $y_i < 1$.

The obtained contradiction implies that p_ϵ are uniformly bounded. Because each p_ϵ is convex, the family is equi-continuous, and, as before, we can use the Arzelà-Ascoli theorem to conclude that a subsequence of p_ϵ converges to some convex continuous p .

A.12.3 Proof of Lemma 7

First, we state a multidimensional generalization of the Intermediate Value Theorem (equivalent to Brouwer's Fixed Point Theorem).

Theorem (Poincaré–Miranda). *Let f_i , $i = 1, \dots, n$, be continuous functions of n variables on $[-1, 1]^n$. Assume that $f_i < 0$ when $x_i = -1$ and $f_i > 0$ when $x_i = +1$, for all i . Then there exists a point in $[-1, 1]^n$ where all f_i are simultaneously 0.*

Consider the rectangular region $(a, b) \in [\frac{1+\alpha}{\beta}, M] \times [0, M]$.²⁴ $M \gg 0$ is determined as follows. A_1 is a subset of the set $\{(x, y) : x \geq \beta - \delta_b\}$ (because $(x, y) \in A_1$ requires $p_1(x, y) = bx + 1 - b\beta \geq p_0(x, y) = 0$), where δ_b can be made arbitrarily small by taking $b \rightarrow \infty$. Furthermore, for sufficiently large b , for any a , A_1 contains a region of positive mass where $x \geq \beta$ (in particular, it contains the region $\{(x, y) : x \geq \beta + \varepsilon_b, x + y \leq 2\beta - \varepsilon_b\}$, where $\varepsilon_b \rightarrow 0$ as $b \rightarrow \infty$, because for all (x, y) in that set $p_1(x, y) \rightarrow \infty$ and $p_{12}(x, y) < 2 + \alpha$). Since F is continuous, the mass contained in the region $\{(x, y) : \beta - \delta_b \leq x < \beta\}$ goes to 0 as $\delta_b \rightarrow 0$. Thus, there exists an M_b such that for all $N \geq M_b$, $t(a, N) > \beta$, for all a . Next, note that if $a \geq M_b$, A_{12} contains the square $\{(x, y) : x \geq \beta, y \geq \beta\}$, which has positive mass. Also, A_{12} is a subset of the triangle $\{(x, y) : x + y \geq 2\beta - \delta_a\}$, where δ_a can be made arbitrarily small by taking $a \rightarrow \infty$. Finally, A_{12} is symmetric so its center of mass lies on the $x = y$ diagonal. As $a \rightarrow \infty$ the mass of the region $\{(x, y) : 2\beta - \delta_a \leq x + y < 2\beta\}$ goes to 0. Thus, there exists an $M \geq M_b$ such that $s(M, b) > \beta$, for all $b \leq M$.

First, we check that s and t are defined on $[\frac{1+\alpha}{\beta}, M] \times [0, M]$. It is enough to show that a neighborhood of (β, β) is contained in A_{12} , and likewise of $(\beta, 0) \in A_1$, respectively; as long as the

²⁴When $b = 0$, regions A_1 and A_2 merge, since $p_1 \equiv p_2 \equiv 1$. This can be ruled out by imposing a small but positive lower bound on b , but it is more convenient instead to allow $b = 0$ and separate A_1 and A_2 along the $x = y$ diagonal.

regions are non-empty, s and t are well-defined. First, for any a and b , $p_{12}(\beta, \beta) = 2 + \alpha > 1$, while $p_1(\beta, \beta) = p_2(\beta, \beta) = 1$. Therefore $p_{12}(\beta, \beta) = p(\beta, \beta)$ and since these are affine planes, some neighborhood of (β, β) is contained in A_{12} . Second, for any b , $p_{12}(\beta, 0) = 2 + \alpha - a\beta$. Hence, for all $a > \frac{1+\alpha}{\beta}$, $p_{12}(\beta, 0) \leq 1 = p_1(\beta, 0)$ hold. As before, some neighborhood of $(\beta, 0)$ is contained in A_1 (and, by symmetry, the same holds for $(0, \beta)$ and A_2). Finally, if $a = \frac{1+\alpha}{\beta}$, A_1 may have empty interior; however, as shown in the next paragraph, we can extend t by continuity.

Next, we check that s and t are continuous in (a, b) . We first prove that the integral $\int_{A_{12}(a,b)} xf(x, y) dx dy$ is continuous in (a, b) , using continuity along sequences. Let $(a_n, b_n) \rightarrow (a, b)$. The sequence of indicator functions $\mathbb{1}_{A_{12}(a_n, b_n)}$ converges pointwise to $\mathbb{1}_{A_{12}(a, b)}$ almost everywhere (with respect to the Lebesgue measure on $[0, 1]^2$). To see this, note that $\mathbb{1}_{A_{12}(a_n, b_n)}(x, y) = 1$ if and only if $p_{12}(x, y; a_n, b_n) \geq p_1(x, y; a_n, b_n)$, $p_{12}(x, y; a_n, b_n) \geq p_2(x, y; a_n, b_n)$, and $p_{12}(x, y; a_n, b_n) \geq p_\emptyset(x, y; a_n, b_n)$ (and is 0 otherwise). For any $(x, y) \in [0, 1]^2$, the inequalities are linear, in particular continuous, in (a_n, b_n) . If at least one fails for (a, b) , then it also fails for (a_n, b_n) for sufficiently large n . If they hold *strictly* for (a, b) , then they hold weakly in the limit of (a_n, b_n) . Finally, if all hold at (a, b) , at least one with equality, then pointwise convergence cannot be guaranteed; however, the set of (x, y) where this occurs has measure zero, since the inequalities are linear in (x, y) . Having established pointwise convergence almost everywhere, note that $xf(x, y)\mathbb{1}_{A_{12}(a_n, b_n)}$ is bounded above by $\max f(x, y)$, which we assumed to be finite. Then, by the dominated convergence theorem,

$$\int_{A_{12}(a_n, b_n)} xf(x, y) dx dy = \int_{[0, 1]^2} xf(x, y) \mathbb{1}_{A_{12}(a_n, b_n)} dx dy \rightarrow \int_{[0, 1]^2} xf(x, y) \mathbb{1}_{A_{12}(a, b)} dx dy$$

which proves continuity of $\int_{A_{12}(a_n, b_n)} xf(x, y) dx dy$. The argument for $\int_{A_{12}(a, b)} f(x, y) dx dy$ is identical. Therefore $s(a, b)$, the ratio of the two preceding integrals, is continuous as long as the denominator does not go to 0. This can only occur on the boundary of $[\frac{1+\alpha}{\beta}, M] \times [0, M]$, specifically if $a = \frac{1+\alpha}{\beta}$; since s is bounded (to $[0, 1]$), it can be extended by continuity on the boundary. This establishes continuity of s ; the argument for t is identical.

We will now verify that $s(a, b) - \beta$ and $t(a, b) - \beta$ satisfy the required inequalities on the sides of the rectangular region. By rescaling, the Poincaré–Miranda Theorem guarantees the existence of (a^*, b^*) such that $s(a^*, b^*) = \beta$ and $t(a^*, b^*) = \beta$.

When $b = 0$, for any $a \in [\frac{1+\alpha}{\beta}, M]$, A_1 is equal to set $\{(x, y) \in [0, 1]^2 : x \geq y, x + y \leq \underline{s}\}$, for some $\underline{s} < \beta$ which depends on a . By assumption (1) then, $t(a, 0) < \beta$ for all $a \in [\frac{1+\alpha}{\beta}, M]$.

When $a = \frac{1+\alpha}{\beta}$, there are two cases to consider. When $b \leq \frac{1+\alpha}{\beta}$ (so that $p_{12} \geq p_1$ for all $x \geq \beta$), the set A_{12} contains $\{(x, y) : x + y \geq \beta\}$, and adding points with $x + y < \beta$ only lowers its center of mass. By assumption (2) then, $s(\frac{1+\alpha}{\beta}, b) < \beta$ for $b \leq \frac{1+\alpha}{\beta}$.

When instead $b > \frac{1+\alpha}{\beta}$, the set A_{12} can be decomposed into two sets: one with (x, y) that satisfy $x + y \leq 2\beta$, and one where $x + y \geq 2\beta$. The former set contains $\{(x, y) : x + y \geq \frac{\alpha\beta}{1+\alpha}, x \leq \beta, y \leq \beta\} \subseteq \{(x, y) : x + y \geq \beta, x \leq \beta, y \leq \beta\}$. The latter set is contained by $\{(x, y) : x + y \geq 2\beta\}$, trivially. Note that adding points to the former lowers the center of mass, and so does removing

points from the latter. Thus, by assumption (3), and putting together with the previous paragraph, we conclude that $s(\frac{1+\alpha}{\beta}, b) < \beta$ for all $b \in [0, M]$.

Finally, by choice of M , $s(M, b) > \beta$ for all $b \in [0, M]$ and $t(a, M) > \beta$ for all $a \in [\frac{1+\alpha}{\beta}, M]$.

We conclude that there exist a^* and b^* such that $s(a^*, b^*) = \beta$ and $t(a^*, b^*) = \beta$.

A.12.4 Proof of Lemma 8

Take an arbitrary $\mu \in \Delta(\Theta)$. To simplify the formulas, with slight abuse of notation, we will write $u_A(x)$ instead of $u_A(\beta x)$. The outer-convexity condition is

$$\int_{\underline{x}}^{\bar{x}} p(x, u_A(x)) d\mu(x) \geq p(\mathbb{E}_\mu X, \mathbb{E}_\mu u_A(X)).$$

Using the structure of the function p , we can write the left hand side as

$$\int_{\underline{x}}^{\bar{x}} h(x, u_A(x)) d\mu(x) + \int_{\underline{x}}^{x^*} [v(x, u_A(x)) - h(x, u_A(x))] d\mu(x).$$

Because h is affine, we have

$$\int_{\underline{x}}^{\bar{x}} h(x, u_A(x)) d\mu(x) = h(\mathbb{E}_\mu X, \mathbb{E}_\mu u_A(X)),$$

so it is enough to prove that

$$\int_{\underline{x}}^{x^*} [v(x, u_A(x)) - h(x, u_A(x))] d\mu(x) \geq p(\mathbb{E}_\mu X, \mathbb{E}_\mu u_A(X)) - h(\mathbb{E}_\mu X, \mathbb{E}_\mu u_A(X)).$$

In the case $\mathbb{E}_\mu X \geq x^*$, the right hand side is equal to zero, so the inequality holds trivially (the integrand on the left hand side is non-negative). From now on, we consider the case $\mathbb{E}_\mu X < x^*$. Using the fact that $v(x, y) - h(x, y) = (1 - \beta)(y - 2)(x - x^*)/2$, we can write the inequality as

$$\int_{\underline{x}}^{x^*} (u_A(x) - 2)(x - x^*) d\mu(x) \geq (\mathbb{E}_\mu u_A(X) - 2)(\mathbb{E}_\mu X - x^*).$$

Rewriting $\mathbb{E}_\mu u_A(X)$ on the right hand side as $\int_{\underline{x}}^{x^*} u_A(x) d\mu(x) + \int_{x^*}^{\bar{x}} u_A(x) d\mu(x)$, and rearranging, we obtain

$$\int_{\underline{x}}^{x^*} u_A(x)(x - \mathbb{E}_\mu X) d\mu(x) - 2 \int_{\underline{x}}^{x^*} (x - x^*) d\mu(x) \geq \int_{x^*}^{\bar{x}} u_A(x)(\mathbb{E}_\mu X - x^*) d\mu(x) - 2(\mathbb{E}_\mu X - x^*).$$

Simplifying further, we obtain

$$\int_{\underline{x}}^{x^*} u_A(x)(x - \mathbb{E}_\mu X) d\mu(x) - 2\mu(x^*)\mathbb{E}_\mu[X|X \leq x^*] + 2x^*\mu(x^*) \geq (x^* - \mathbb{E}_\mu X) \left[2 - \int_{x^*}^{\bar{x}} u_A(x) d\mu(x) \right].$$

The function $u_A(x)(x - \mathbb{E}_\mu X)$ is convex in x (because $u_A(x)x$ is convex, and $u_A(x)$ is concave), so we can use Jensen's Inequality to conclude that it is sufficient to prove that

$$\begin{aligned} \mu(x^*)u_A(\mathbb{E}_\mu[X|X \leq x^*]) (\mathbb{E}_\mu[X|X \leq x^*] - \mathbb{E}_\mu X) - 2\mu(x^*)\mathbb{E}_\mu[X|X \leq x^*] + 2x^*\mu(x^*) \\ \geq (x^* - \mathbb{E}_\mu X) \left[2 - \int_{x^*}^{\bar{x}} u_A(x) d\mu(x) \right]. \quad (\text{A.31}) \end{aligned}$$

The left hand side of (A.31) can be written as

$$\mu(x^*)2(x^* - \mathbb{E}_\mu X) + \mu(x^*) (\mathbb{E}_\mu X - \mathbb{E}_\mu[X|X \leq x^*]) (2 - u_A(\mathbb{E}_\mu[X|X \leq x^*])).$$

Simplifying further, we can rewrite (A.31) as

$$\begin{aligned} \mu(x^*) (\mathbb{E}_\mu X - \mathbb{E}_\mu[X|X \leq x^*]) (2 - u_A(\mathbb{E}_\mu[X|X \leq x^*])) \\ \geq (x^* - \mathbb{E}_\mu X)(1 - \mu(x^*)) [2 - \mathbb{E}_\mu [u_A(X)|X \geq x^*]]. \end{aligned}$$

Next, we use the inequality

$$u_A(\mathbb{E}_\mu[X|X \leq x^*]) \leq u_A(x^*) \leq \mathbb{E}_\mu [u_A(X)|X \geq x^*]$$

to notice that it is enough to prove that

$$\mu(x^*) (\mathbb{E}_\mu X - \mathbb{E}_\mu[X|X \leq x^*]) \geq (x^* - \mathbb{E}_\mu X)(1 - \mu(x^*)).$$

Simplifying,

$$\mu(x^*) (x^* - \mathbb{E}_\mu[X|X \leq x^*]) \geq (x^* - \mathbb{E}_\mu X).$$

Finally, we can use the fact that

$$\mathbb{E}_\mu X = \mu(x^*)\mathbb{E}_\mu[X|X \leq x^*] + (1 - \mu(x^*))\mathbb{E}_\mu[X|X \geq x^*]$$

to rewrite the above inequality (after some simplifications) as

$$\mathbb{E}_\mu[X|X \geq x^*] \geq x^*.$$

The last inequality is obviously true which ends the proof.

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